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Calcul pseudodifférentiel sur les variétés filtrées et applications à la géométrie et l'analyse harmonique

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Abstract

This thesis is devoted to three different questions tied to the pseudodifferential calculus on filtered manifolds and their applications.

The first chapter presents is a detailed introduction. The second consists of preliminaries on groupoids, filtered manifolds and their pseudodifferential calculus and representations of nilpotent Lie groups.

In chapter 3 we extend a construction of Debord and Skandalis to the filtered case. As a corrolary we obtain a decomposition of the symbol algebra into a sequence of nested ideals with subquotients isomorphic to continuous functions on locally compact spaces with value in algebras of compact operators. This extends a result of Epstein and Melrose for contact manifolds.

In chapter 4 we introduce a notion of foliated filtered manifolds they are foliated manifolds with a structure of filtered manifold on the leaf space. We use this setup to define a transversal Rockland condition for operators in the filtered calculus. For operators satisfying this condition we construct several classes in KK-theory and show Poincaré duality type relations between them.

In chapter 5 we push these ideas further to define transversal Bernstein-Gelfand-Gelfand sequences for foliated manifolds with a transverse parabolic geometry in the sense of Cartan. We show that these sequences satisfy a transversal Rockland condition in a graded sense, as well as the transverse twisted de Rham complex used to construct the BGG sequence. In the flat case we show the isomorphism between the cohomology of the two complexes.

Résumé

Cette thèse présente l'étude de trois différentes questions liées au calcul pseudodifférentiel sur les variétés filtrées et ses applications.

Le premier chapitre est une introduction. Le deuxième consiste en des préliminaires sur les groupoïdes, les variétés filtrées et leur calcul pseudodifférentiel et les représentations des groupes de Lie nilpotents.

Le chapitre 3 présente l'extension d'une construction de Debord et Skandalis aux variétés filtrées. Comme corollaire de la preuve nous obtenons une décomposition de l'algèbre des symboles en une suite d'idéaux imbriqués dont les sous-quotients sont isomorphes à des algèbres de fonctions continues sur des espaces localement compacts à valeurs dans des algèbres d'opérateurs compacts. Ce résultat étend celui d'Epstein et Melrose sur les variétés de contact.

Dans le chapitre 4 nous introduisons la notion de variété feuilletée filtrée, ce sont des variétés feuilletées dont l'espace des feuilles possède une structure de variété filtrée. Nous définissons dans ce contexte une condition de Rockland transverse. A un opérateur pseudodifférentiel satisfaisant cette condition nous associons plusieurs classes de KK-théorie et montrons une relation de dualité de Poincaré entre ces classes.

Le chapitre 5 pousse ces idées plus loin dans le but de définir une suite de Bernstein-Gelfand-Gelfand transverse pour les variétés feuilletées avec une géométrie parabolique transverse au sens de Cartan. Nous montrons que ces suites d'opérateurs satisfont la condition de Rockland transverse en un sens gradué, de même que le complexe de de Rham transverse tordu utilisé pour construire les suites BGG. Dans le cas plat nous montrons un isomorphisme entre les cohomologies des deux complexes.

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Contents

Résumé détaillé

Cette thèse est dédiée à l'étude du calcul pseudodifférentiel pour les variétés filtrées. Nous y étendons des constructions connues dans le cadre pseudodifférentiel classique et nous intéressons aussi à des variétés feuilletées dont l'espace des feuilles est filtré.

Un opérateur pseudodifférentiel elliptique sur une variété compacte sans bord s'étend en un opérateur de Fredholm et admet donc un indice analytique (la différence de dimension entre son noyau et conoyau). Gelfand a remarqué le premier [39], au vu d'exemples donnés par les résultats d'Hirzebruch sur les théorèmes de Riemann-Roch et de signature (voir [45]), l'invariance de cette indice par homotopie. Cette observation l'a amené à chercher une méthode topologique pour calculer l'indice. La réponse fut apportée par Atiyah et Singer dans leur célèbre article [5] avec des méthodes de calcul issues de la topologie algébrique. L'idée fut généralisée par Connes et Skandalis [22] pour les opérateurs longitudinalement elliptiques sur les feuilletages. Dans ce contexte, l'indice n'est plus un nombre mais une classe de K-théorie de la C^*-algèbre du feuilletage.

Il existe cependant des opérateurs qui, bien que non elliptiques, sont Fredholm et ont donc un indice analytique. Historiquement de tels opérateurs ont été considéré par Hörmander avec la somme des carrés de champs de vecteurs [47], par Folland et Stein avec le laplacien de Kohn sur les variétés fortement pseudoconvexes ou encore par Beals-Greiner [8] et van Erp [78, 79] avec le sous-laplacien sur les variétés de contact et variétés de Heisenberg. L'idée commune dans ces exemples est l'existence d'un sous-fibré $H \subset TM$ nous permettant de considérer les champs de vecteurs le long de H comme des opérateurs d'ordre 1 et les autres comme d'ordre 2. Par exemple si M est une variété de contact, H la distribution d'hyperplans, les sous-laplaciens :

$$\sum_{i=1}^{2n} X_i^2 + \gamma T$$

où T désigne le champ de Reeb, $(X_i)_{1 \leq i \leq 2n}$ est une base (locale) de H et $\gamma \in \mathbb{C}$ ne peuvent pas être elliptiques. Pour certaines valeurs de γ ils sont en revanche hypoelliptiques et même Fredholm. Le symbole principal ne prend ici pas en compte la direction du champ de Reeb, il est donc naturel de le considérer comme un opérateur d'ordre 2 pour le prendre en compte dans le symbole principal.

Connes a montré l'importance du groupoïde tangent [21] dans l'étude des problèmes d'indice. Ce groupoïde peut être défini algébriquement par :

$$\mathbb{T}M = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

Il possède une structure lisse naturelle issue de la construction de déformation au cône normal. Ce groupoïde a été généralisé au cas des variétés de contact par van Erp [78] et aux variétés de Heisenberg par Ponge [70]. Plus récemment la construction a été généralisée aux variétés filtrées générales [80, 18, 63] et le calcul pseudodifférentiel adapté à ces variétés a été construit via cette approche [81]. Plus précisément, une variété M est dite filtrée elle possède une filtration de son fibré tangent par des sous-fibrés :

$$\{0\} = H^0 \subset H^1 \subset \cdots \subset H^r = TM,$$

vérifiant la condition sur le crochet de Lie des sections :

$$\forall i, j, \left[\Gamma(H^i), \Gamma(H^j)\right] \subset \Gamma(H^{i+j}).$$

Dans ce contexte on peut considérer les sections de H^i comme des opérateurs d'ordre i, la condition sur le crochet de Lie rendant cette nouvelle notion d'ordre compatible avec la composition des opérateurs. Cette filtration produit naturellement une algébroïde :

$$\mathfrak{t}_H M = \bigoplus_{i=1}^{r} H^i \diagup H^{i-1}.$$

Cette algébroïde est en fait un champ d'algèbres de Lie (nilpotentes graduées) que l'on peut intégrer en un groupoïde $T_H M$ qui est un champ de groupes de Lie nilpotents gradués. Ce groupoïde remplace le fibré tangent dans le calcul classique. L'analogue du groupoïde tangent devient donc, au niveau algébrique:

$$\mathbb{T}_H M := M \times M \times \mathbb{R}^* \sqcup T_H M \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

Le calcul symbolique se fait alors via des opérateurs de convolution sur le groupoïde $T_H M$ et est ainsi non-commutatif. La condition d'ellipticité est remplacée par la condition de Rockland [73, 19] portant sur l'image du symbole dans les représentations irréductibles non-triviales du groupoïde $T_H M$.

On peut alors montrer que les opérateurs dont le symbole satisfait cette condition sont de Fredholm. Le problème de trouver une formule de l'indice dans ce contexte a été résolu par Mohsen [64] suite à des travaux de Baum et van Erp dans le cas des variétés de contact [7].

Cette thèse présente les contributions de l'auteur dans ces directions. Elle est composée de quatre autres chapitres. Le premier est une exposition préliminaires des outils et résultats utiles pour la suite. Nous y rappelons les bases sur les groupoïdes, leurs algèbres de convolution ainsi que la construction et les propriétés du calcul pseudodifférentiel tel que construit dans [81]. La condition de Rockland impliquant les représentations des groupes de Lie nilpotents, nous faisons aussi des rappels sur la méthode des orbites de Kirillov et les stratifications du dual unitaire qu'elles impliquent (d'après Pedersen). Nous présentons dans cette section une construction originale de l'algèbre de Schwartz du groupoïde tangent dans le cas filtré. Cette construction suit les idées de Carrillo Rouse [15]. Cette algèbre a aussi été construite par Ewert [35] mais sa méthode repose sur les coordonnées du groupoïde tangent introduites par Choi et Ponge [18] là où nous utilisons celles de van Erp et Yuncken [80] (les deux algèbres sont néanmoins les mêmes). Les trois chapitres restants consistent en trois travaux ayant donné lieu à trois articles distincts (un soumis au moment de l'écriture de ces lignes et les deux autres le seront peu de temps après).

Dans le chapitre 3, nous adaptons au cas des variétés filtrées une construction de Debord et Skandalis [30]. Ils ont montré, pour le calcul pseudodifférentiel classique, un isomorphisme entre la clôture C^*-algébrique naturelle de l'algèbre des opérateurs pseudodifférentiels d'ordre 0 et une algèbre construite à partir du groupoïde tangent et de son action de zoom. Restreignons le groupoïde tangent à $\mathbb{T}^+M \rightrightarrows M \times \mathbb{R}_+$. On a la suite exacte :

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C^*(\mathbb{T}^+M) \longrightarrow \mathscr{C}_0(T^*M) \longrightarrow 0.$$

L'action de zoom $(\alpha_\lambda)_{\lambda>0}$ est définie sur $\mathbb{T}^+M$ via :

$$\begin{aligned} \alpha_\lambda(x,y,t) &= (x,y,\lambda^{-1}t) & (x,y,t) &\in M \times M \times \mathbb{R}_+^* \\ \alpha_\lambda(x,\xi,0) &= (x,\lambda\xi,0) & (x,\xi) &\in TM. \end{aligned}$$

Si on enlève la section nulle du fibré cotangent on obtient la suite exacte :

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C_0^*(\mathbb{T}^+M) \longrightarrow \mathscr{C}_0(T^*M \setminus \{0\}) \longrightarrow 0.$$

L'action de zoom est alors libre et propre et on a :

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \longrightarrow C_0^*(\mathbb{T}^+M) \rtimes_\alpha \mathbb{R}_+^* \longrightarrow \mathscr{C}(\mathbb{S}^*M) \otimes \mathcal{K} \longrightarrow 0.$$

Debord et Skandalis ont relié cette suite exacte à celle des opérateurs pseudodifférentiels d'ordre 0 :

$$0 \longrightarrow \mathcal{K} \longrightarrow \Psi^*(M) \longrightarrow \mathscr{C}(\mathbb{S}^*M) \longrightarrow 0,$$

par une équivalence de Morita explicite, la flèche de droite désignant ici le symbole principal. Une étape importante dans leur article est le fait de montrer que tous les opérateurs pseudodifférentiels s'écrivent sous la forme

$$\int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t},$$

où $(f_t)_{t\geq 0}$ est une fonction dans l'idéal $\mathcal{J}^+(M)$ des fonctions dans la classe de Schwartz de $\mathbb{T}^+M$ dont la transformée de Fourier est infiniment plate sur la section nulle de T^*M. Pour les variétés filtrées une construction similaire des opérateurs pseudodifférentiels est possible. Nous montrons que les opérateurs qui apparaissent avec de telles intégrales sont les mêmes que ceux définis par van Erp et Yuncken dans [81]. Plus précisément définissons l'idéal :

$$\mathcal{J}_H^+(M) \subset \mathscr{S}(\mathbb{T}_H^+M),$$

et nous montrons le résultat suivant :

Theorem 0.0.1. *Soit (M, H), une variété filtrée, $m \in \mathbb{C}$ et $\Psi_H^m(M)$ la classe des opérateurs pseudodifférentiels filtrés d'ordre m au sens de [81]. Si $(f_t)_{t\geq 0} \in \mathcal{J}_H^+(M)$ alors*

$$\int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t} \in \Psi_H^m(M).$$

De plus le symbole principal de cet opérateur est donné par $\int_0^{+\infty} t^{-m} \delta_{\lambda *} f_0 \frac{\mathrm{d}t}{t}$ [1]. *Réciproquement tout opérateur pseudodifférentiel dans $\Psi_H^m(M)$ s'écrit sous cette forme.*

En utilisant ce résultat on peut définir un morphisme de C^*-algèbres :

$$\Psi_H^*(M) \rtimes \mathbb{R} \to C_0^*(\mathbb{T}_H^+M)$$ [2],

cette dernière algèbre étant obtenu en prenant la pré-image dans la suite exacte du groupoïde tangent de l'algèbre $C_0^*(T_HM)$ correspondant au noyau

[1] $(\delta_\lambda)_{\lambda>0}$ est la famille de dilatation inhomogène sur les fibres de T_HM, c'est le prolongement naturel de l'action de zoom sur $\mathbb{T}_HM$.

[2] L'action de $\mathbb{R}$ est la conjugaison par des puissances complexes d'un opérateur bien choisi.

des représentations triviales de chaque fibre. Cette application est $\mathbb{R}_+^*$-équivariante pour l'action duale au départ et l'action de zoom à l'arrivée. Pour montrer qu'il s'agit d'un isomorphisme Debord et Skandalis utilisent les suites exactes respectives des deux algèbres et un lemme de deux sur trois. Ici en revanche le fait que notre application est un isomorphisme au niveau des quotients respectifs n'est plus évident du tout car l'action sur les symboles principaux n'est plus triviale. Nous dédions donc une section du chapitre au cas des symboles et nous ramenons à un problème d'analyse harmonique sur les groupes nilpotents gradués. Pour cela nous utilisons la théorie de Kirillov et la stratification de Pedersen des orbites co-adjointes pour décomposer la C^*-algèbre des symboles d'ordre 0, $\Sigma(G)$, en une suite d'idéaux imbriqués. Nous obtenons le résultat suivant :

Theorem 0.0.2. *Soit G un groupe de Lie nilpotent gradué, il existe une suite d'idéaux imbriqués :*

$$0 = \Sigma_0(G) \triangleleft \Sigma_1(G) \triangleleft \cdots \triangleleft \Sigma_r(G) = \Sigma(G),$$

telle que pour tout $i \geq 0$ on ait $\Sigma_i(G)/\Sigma_{i-1}(G) \cong \mathscr{C}_0\left(\Lambda_i/\mathbb{R}_+^*, \mathcal{K}_i\right)$.

Ici $\Lambda_i \subset \mathfrak{g}^*/G$ est le quotient d'un cône algébrique. C'est un espace séparé et $\Lambda_i/\mathbb{R}_+^*$ est localement compact séparé. L'algèbre $\mathcal{K}_i$ désigne l'algèbre des opérateurs compacts sur un espace de Hilbert séparable, de dimension infinie pour $i < r$ et de dimension 1 pour $i = r$.

Cette décomposition généralise celle d'Epstein et Melrose [34] pour le groupe d'Heisenberg et les variétés de contact[3]. Ces deux constructions permettent de montrer :

Theorem 0.0.3. *L'application $\Psi_H^*(M) \rtimes \mathbb{R} \to C_0^*(\mathbb{T}_H^+ M)$ est un isomorphisme. En particulier on obtient l'équivalence de Morita :*

$$\Psi_H^*(M) \sim C_0^*(\mathbb{T}_H^+ M) \rtimes \mathbb{R}_+^*.$$

Il est à noter que l'équivalence de Morita a été obtenue par différentes méthodes dans [35].

Dans le chapitre 4 nous adaptons la notion d'ellipticité transverse au cas filtré et montrons un théorème d'indice transverse. Nous introduisons la notion de variété feuilletée filtrée, il s'agit de variétés feuilletées dont l'espace

[3]Le théorème s'adapte immédiatement à $T_H M$ si il s'agit d'un fibré en groupes localement trivial.

des feuilles est une filtré. Plus précisément c'est une variété M munie d'une filtration du fibré tangent :

$$H^0 \subset H^1 \subset \cdots \subset H^r = TM,$$

avec une condition sur les crochets de Lie de sections similaires à celle des variétés filtrées. En particulier H^0 est un feuilletage. Le fibré transverse est filtré et on peut définir le gradué associé $T_{H/H^0}M$ comme un quotient du groupoïde $T_H M$ (cela permet par poussée en avant de définir une restriction transverse des symboles). Le groupoïde d'holonomie de H^0 agit sur le groupoïde $T_{H/H^0}M$ ce qui permet de définir une condition de Rockland transverse. Nous construisons alors un groupoïde de déformation adapté aux problèmes transverses, algébriquement il s'écrit :

$$\mathbb{T}_H^{\mathrm{Hol}} M := M \times M \times \mathbb{R}^* \sqcup \mathrm{Hol}(H^0) \ltimes T_{H/H^0}M \times \{0\}.$$

Ce groupoïde déforme uniquement dans les directions transverses. Plus précisément localement on peut se ramener au cas d'un feuilletage produit $L \times T$ avec T une variété feuilletée. Le groupoïde de déformation usuel devient :

$$\mathbb{T}^{\mathrm{Hol}}_{\{0\}\times H}(L \times T) = \mathbb{T}L \times_{\mathbb{R}} \mathbb{T}_H T,$$

et le groupoïde des déformations transverses devient :

$$\mathbb{T}_{\{0\}\times H}(L \times T) = L \times L \times \mathbb{T}_H T = \mathrm{Hol}(TL \times \{0\}) \times_T \mathbb{T}_H T.$$

Pour construire un cycle de K-homologie à partir d'un opérateur dont le symbole est transversalement Rockland il faut d'abord l'étendre en un noyau sur $\mathbb{T}_H M$ en suivant la philosophie de [81] puis le pousser en avant sur $\mathbb{T}_H^{\mathrm{Hol}} M$. Cette opération créée des singularités qu'il convient d'analyser correctement en coordonnées locales afin de définir le cycle transverse. Cette méthode et les constructions générales de classes de KK-théorie à partir de groupoïdes de déformation (voir [31] par exemple) fait le lien entre la classe de KK-théorie construite via le symbole et celle de K-homologie donnée par l'opérateur. Les résultats de ce chapitre peuvent être résumés dans le théorème suivant :

Theorem 0.0.4. *Soit (M, H) une variété filtrée feuilletée, $E \to M$ un fibré hermitien ${}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$-gradué, $\mathrm{Hol}(H^0)$-équivariant. Soit $\sigma \in \Sigma^0(T_{H/H^0}M, E)$ un symbole transverse d'ordre 0, $\mathrm{Hol}(H^0)$-invariant tel que $\sigma^2 = 1$.*

Soit $\mathbb{P} \in \Psi^*_H(M, E)$ avec $\int_{H^0} \mathbb{P}_0$ représentant σ et posons $P = \mathbb{P}_1$. On a alors :

- *σ induit une classe $[\sigma] \in KK^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$*

- *P induit une classe $[P] \in KK(C^*(\mathrm{Hol}(H^0), \mathbb{C})$ ne dépendant que de σ*
- *$\mathbb{P}$ induit une classe $[\mathbb{P}] \in KK(C^*(\mathrm{Hol}(H^0)), C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]}))$ ne dépendant que de σ*
- *on a les relations $\mathrm{ev}_0([\mathbb{P}]) = j_{\mathrm{Hol}(H^0)}([\sigma])$ où $j_{\mathrm{Hol}(H^0)}$ est le morphisme de descente et $\mathrm{ev}_1([\mathbb{P}]) = [P]$*

En particulier si H^0 est un feuilletage moyennable, on obtient la dualité de Poincaré :

$$[P] = \mathrm{Ind}_H^{hol} \otimes j_{\mathrm{Hol}(H^0)}([\sigma]),$$

où $\mathrm{Ind}_H^{\mathrm{Hol}} = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \in KK(\mathrm{Hol}(H^0) \ltimes T_{H/H^0} M, \mathbb{C})$.

Nous terminons le chapitre par l'étude détaillée de deux exemples particuliers : les feuilletages induits par des fibrations et par des actions de groupes discrets.

Dans le chapitre 5 nous poursuivons l'idée de condition de Rockland transverse dans deux directions. Nous remplaçons les opérateurs par des suites d'opérateurs (par exemples des complexes) et remplaçons la condition de Rockland par la condition de Rockland graduée au sens de [27].

La motivation principale est la création de suites d'opérateurs de type Bernstein-Gelfand-Gelfand pour des variétés feuilletées avec des géométries paraboliques transverses au sens de Cartan. Les suites BGG ont une origine algébrique dans les travaux de Bernstein, Gelfand et Gelfand [10] où elles ont été introduites comme résolutions projectives pour les représentations de plus haut poids de groupes de Lie semi-simples. Suivant l'approche de Kostant du théorème de Borel-Weil-Bott, elles ont pu être interprétées comme résolution de certains faisceaux de fonctions constantes sur des variétés de drapeaux généralisés. Čap, Slovák et Souček ont poussé cette approche plus loin, remarquant que les modules de Verma utilisés dans l'approche initiale de BGG avaient une interprétation comme fibré de jets sur les variétés de drapeaux généralisés. Cette observation leur a permis d'étendre la construction aux variétés ayant une géométrie parabolique au sens de Cartan [16], c'est à dire "localement modelées" sur l'espace homogène G/P où $P \subset G$ est un sous-groupe parabolique.

La suite BGG peut être définie à partir du complexe de de Rham tordu sur un fibré tractoriel associé à une représentation de G et muni de sa connexion tractorielle naturelle. Nous montrons que cette suite d'opérateur est transversalement Rockland graduée en un sens que nous définissons (adaptant la notion de Dave et Haller [27]). Nous montrons que le complexe des symboles transverses est donné par le complexe de Chevalley-Eilenberg pour la cohomologie de l'algèbre de lie $\mathfrak{g}$ et de la représentation utilisée pour construire le fibré tractoriel. Plus précisément nous montrons :

Theorem 0.0.5. *Soit* (M, H) *une variété filtrée feuilletée,* $E \to M$ *un fibré vectoriel gradué et* ∇ *une connection* $\mathrm{Hol}(H^0)$*-équivariante sur* E*. On suppose que la composante de bidegré* $(2, 0)$ *de la courbure de* ∇ *est de degré* 1 *pour les filtrations en jeu*[4]*. La suite d'opérateurs donnée par le complexe de de Rham tordu transverse* $(\Omega^{\bullet,0}(M, E), \mathrm{d}_N^\nabla)$ *est transversallement Rockland graduée.*

Nous définissons dans le cas des géométries de Cartan transverse une suite d'opérateurs de type BGG $(\Gamma(\mathcal{H}_\bullet, D_\bullet)$ à partir de la suite précédente et montrons :

Theorem 0.0.6. *La suite d'opérateurs de Bernstein-Gelfand-Gelfand transverse* $(\Gamma(\mathcal{H}_\bullet), D_\bullet)$ *est transversalement Rockland graduée. Si de plus la suite de de Rham tordue transverse est un complexe alors c'est aussi le cas de la suite BGG transverse et l'application naturelle entre les deux complexes est un morphisme de complexes, inversible à homotopie près. En particulier les deux complexes ont la même cohomologie.*

[4] C'est le cas pour la connection tractorielle.

Chapter 1

Introduction

If a pseudodifferential operator on a closed manifold is elliptic then it extends to a Fredholm operator and has an analytic index (the difference of dimensions between the kernel and cokernel). This was first observed by Gelfand [39] who, motivated by Hirzebruch's proofs of Riemann-Roch and signature theorem (see [45]), proved the homotopy invariance of the index and asked for topological methods of computation. It was Atiyah and Singer [5] who then gave algebraic and topological methods to compute this index. This idea was generalized by Connes and Skandalis [22] for longitudinally elliptic operators on foliations. In this case the index is not a number but a element of the K-theory group of the C^*-algebra of the holonomy groupoid.

On special types of manifolds some operator arised and, although they were not elliptic, they were still Fredholm and thus had an analytical index. Historical examples include Kohn's laplacian on strongly pseudoconvex manifolds studied by Folland and Stein [37] and sublaplacians on contact or Heisenberg manifolds [8, 78, 79]. The common idea in this context is the existence of a subbundle of the tangent bundle $H \subset TM$ so that vector fields in the H directions are to be considered of order 1 and vector fields in transverse directions of order 2. For instance on contact manifolds (where H is a hyperplane distributions) the sublaplacians:

$$\sum_{i=1}^{2n} X_i^2 + \gamma T$$

cannot be ellitpic in the usual sense. Here $(X_i)_{1 \leq i \leq 2n}$ is a (local) basis of H, $\gamma \in \mathbb{C}$ and T is the Reeb field. Its classical symbol forgets everything that happens in the direction of the Reeb field since it is only of order 1. The pseudodifferential calculus has thus been adapted on such manifolds to consider T as an operator of order 2.

To study index problems, Connes highlighted the importance if the tangent groupoid [21], which can algebraically be described as the disjoint union:

$$\mathbb{T}M = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

It is endowed with a smooth structure from the deformation to the normal cone construction. This groupoid has been generalized to the contact case by van Erp [78] and to the Heisenberg case by Ponge [70]. More recently this construction of the tangent groupoid has been generalized to arbitrary filtrations (of any depth) of the tangent bundle [80, 18, 63] and a full development of the pseudodifferential calculus on this context has been performed in [81]. In this calculus, principal symbols of operators are not functions on the co-sphere bundle anymore but some convolution operators on a bundle of nilpotent groups naturally obtained from the filtration (the osculating groupoid). This replaces the notion of ellipticity by the Rockland condition [73, 19] which involves the representation theory of the nilpotent groups appearing in the fibers of the osculating groupoid. The index problem found a formula in this context in the work of Mohsen [64] (and van Erp and Baum's work on the contact case [7]).

This thesis presents the contribution of the author in these directions. It consists of four other chapters, the first being a preliminary exposition of the tools that will be used in the subsequent chapters. These results are not new so the proofs are often not detailed (we point to the references where they can be found instead) unless it serves the general comprehension and introduces other tools used later on. The only original thing is a new construction of the Schwartz algebra for the tangent groupoid in the filtered case (see 2.2.4). This algebra is constructed following ideas of Carrillo Rouse [15]. It was already constructed by Ewert in the filtered case [35] but her method relies on another construction of the tangent groupoid (she uses the construction of [18] while we rely on the one from [80]). The two algebra are the same nonetheless. The three remaining chapters are three separate works resulting in three different articles (one submitted by the time these lines are written, the other two are to be submitted shortly after).

In chapter 3 we adapt the construction of Debord and Skandalis [30] to the filtered case. They showed for the classical calculus an isomorphism between the C^* algebra of pseudodifferential operators (of order 0) on a manifold M and an algebra constructed from the tangent groupoid and its zoom action. Indeed if $\mathbb{T}M$ is the tangent groupoid of M we can restrict it to $\mathbb{T}^+M \rightrightarrows M \times \mathbb{R}_+$. It then sits in the exact sequence:

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C^*(\mathbb{T}^+M) \longrightarrow \mathscr{C}_0(T^*M) \longrightarrow 0.$$

Now the zoom action $(\alpha_\lambda)_{\lambda>0}$ is defined on $\mathbb{T}^+M$ via:

$$\begin{aligned} \alpha_\lambda(x,y,t) &= (x,y,\lambda^{-1}t) && (x,y,t)\in M\times M\times \mathbb{R}_+^* \\ \alpha_\lambda(x,\xi,0) &= (x,\lambda\xi,0) && (x,\xi)\in TM. \end{aligned}$$

If we remove the zero section from the fiber at zero we get the exact sequence

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C_0^*(\mathbb{T}^+M) \longrightarrow \mathscr{C}_0(T^*M \setminus \{0\}) \longrightarrow 0.$$

Now the zoom action becomes free and proper. Taking the cross product yields the exact sequence:

$$0 \longrightarrow \mathcal{K}\otimes\mathcal{K} \longrightarrow C_0^*(\mathbb{T}^+M)\rtimes_\alpha \mathbb{R}_+^* \longrightarrow \mathscr{C}(\mathbb{S}^*M)\otimes\mathcal{K} \longrightarrow 0.$$

Debord and Skandalis related this sequence to the one of pseudodifferential operators:

$$0 \longrightarrow \mathcal{K} \longrightarrow \Psi^*(M) \longrightarrow \mathscr{C}(\mathbb{S}^*M) \longrightarrow 0$$

where $\Psi^*(M)$ is the closure of order 0 operators in the algebra of bounded operators on $L^2(M)$ and the arrow on the right is the principal symbol map. As an important step, they showed that any pseudodifferential operator of order m could be written as an integral of the form:

$$\int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t}$$

where $(f_t)_{t\geq 0}$ is a function in a suitable ideal of the Schwartz algebra of the tangent groupoid. In the context of filtered manifold one could perform the same type of construction but this time, the identification at the level of symbols is not trivial at all. To do this we use Kirrilov's theory and Pedersen's stratification of the coadjoint orbits to decompose the algebra of principal symbols of order 0 $\Sigma(G)$ on a graded Lie group G into a finite sequence of nested ideals:

$$0 = \Sigma_0(G) \triangleleft \Sigma_1(G) \triangleleft \cdots \triangleleft \Sigma_r(G) = \Sigma(G),$$

such that for all $i \geq 0$ we have ${}^{\Sigma_i(G)}\!\big/_{\Sigma_{i-1}(G)} \cong \mathscr{C}_0\left({}^{\Lambda_i}\!\big/_{\mathbb{R}_+^*}, \mathcal{K}_i\right)$. Here Λ_i is a certain subset of the set of coadjoint orbits ${}^{\mathfrak{g}^*}\!\big/_{G}$ so that ${}^{\Lambda_i}\!\big/_{\mathbb{R}_+^*}$ is locally compact Haussdorff and $\mathcal{K}_i$ is an algebra of compact operators. This decomposition generalizes the decomposition of symbols of Epstein and Melrose in the contact case (i.e. for the Heisenberg group) [34]. Combining this with the construction on the tangent groupoid gives an isomorphism:

$$\Psi_H^*(M)\rtimes\mathbb{R} \cong C_0^*(\mathbb{T}_H^+M).$$

Here $\mathbb{T}_H^+ M$ is the restriction over $M \times \mathbb{R}_+$ of the tangent groupoid of the filtered manifold (M, H) and the $\mathbb{R}$ action on the C^*-algebra of order 0 pseudodifferential operators in the filtered calculus is constructed so that the isomorphism is $\mathbb{R}_+^*$-equivariant (for the dual action on the left and the zoom action on the right). The majority of the results in this chapter were also obtained by Ewert in [35] using generalized fixed point algebras methods. Her version of the calculus also differs from the one used here (although the C^*-completions of non-positive order operators are the same). We directly show the equivalence of the Van Erp-Yuncken approach and the Debord-Skandalis approach of pseudodifferential calculus. We also use a different decomposition of the symbol algebra than the one in [35] giving a nicer description of the subquotients in the symbol algebra. Finally we show an isomorphism $\Psi_H^*(M) \rtimes \mathbb{R} \cong C_0^*(\mathbb{T}_H^+ M)$ which is stronger than the Morita equivalence $C_0^*(\mathbb{T}_H^+ M) \rtimes \mathbb{R}_+^* \sim \Psi_H^*(M)$ obtained in [35].

In chapter 4 we adapt the notion of transversal ellipticity and the construction of the transversal index of [44] to the filtered setting. We define what we call foliated filtered manifolds and a transversal Rockland condition. We then construct several KK-classes associated to an operator whose principal symbol satisfies this condition and study the relations between them. To do this we show that in our setting, the bundle H^0 defining the foliation is in the center of the osculating algebroid, allowing to define a transversal osculating bundle $T_{H/H^0} M$ (a quotient of the usual osculating bundle). We also construct a deformation groupoid more suited to transverse problems algebraically defined as:

$$\mathbb{T}_H^{\mathrm{Hol}} M := M \times M \times \mathbb{R}^* \sqcup \mathrm{Hol}(H^0) \ltimes T_{H/H^0} M \times \{0\}.$$

This groupoid only deforms in the transverse directions. More precisely if the foliation is given by a product $L \times T$ then T is a filtered manifold in the regular sense, the usual deformation groupoid becomes:

$$\mathbb{T}_{\{0\} \times H}(L \times T) = \mathbb{T}L \times_{\mathbb{R}} \mathbb{T}_H T,$$

while our groupoid becomes:

$$\mathbb{T}_{\{0\} \times H}^{\mathrm{Hol}}(L \times T) = L \times L \times \mathbb{T}_H T = \mathrm{Hol}(TL \times \{0\}) \times_T \mathbb{T}_H T.$$

The idea to construct a K-homology cycle from a transversally elliptic operator is to extend it to $\mathbb{T}_H M$ following van Erp-Yuncken construction of the pseudodifferential calculus. We then push it forward to our deformation groupoid $\mathbb{T}_H^{\mathrm{Hol}} M$. This push-forward however creates singularities non-conormal to the units and one has to do a careful analysis in local charts to define the transverse cycle. This methods, combined with the general KK-class constructed from deformation groupoids (see [31]), gives the link between the KK-class of the symbol and the K-homology class of the operator. More precisely let $P \in \Psi^0_H(M)$ be a pseudodifferential operator of order 0 in the filtered calculus. We extend it to a quasi-homogeneous operator $\mathbb{P}$ on $\mathbb{T}_H M$ and study its push-forward on $\mathbb{T}_H^{\mathrm{Hol}} M$. We then get:

- an equivariant KK-class $[\int_{H^0} \mathbb{P}_0] \in \mathrm{KK}^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$,
- a K-homology class $[P] \in \mathrm{KK}(C^*(\mathrm{Hol}(H^0), \mathbb{C})$,
- a class $[\mathbb{P}] \in \mathrm{KK}(C^*(\mathrm{Hol}(H^0)), C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]}))$

and the relations :

$$[\mathbb{P}] \otimes [\mathrm{ev}_0] = j_{\mathrm{Hol}(H^0)}([\int_{H^0} \mathbb{P}_0]), [\mathbb{P}] \otimes [\mathrm{ev}_1] = [P].$$

Here $j_{\mathrm{Hol}(H^0)}$ is the descent homomorphism and $\mathrm{ev}_0, \mathrm{ev}_1$ are the evaluation map at times 0 and 1 of the deformation groupoid $\mathbb{T}_H^{\mathrm{Hol}} M$.

We end the chapter by the detailed study of two particular examples: foliations given by fibrations and the ones arising from actions of discrete groups.

In chapter 5 we pursue the ideas of the transversal Rockland condition in two directions: we replace operators by complexes and Rockland condition by the graded Rockland condition in the sense of [27]. The motivation is the creation of a Bernstein-Gelfand-Gelfand type complex for foliated manifolds with a transverse parabolic geometry. The BGG sequences have an algebraic origin in the work of Bernstein, Gelfand and Gelfand [10] where it served as a projective resolution for representations of highest weight of semisimple Lie groups. Using Kostant's approach to the Borel-Weil-Bott theorem [54] they were reinterpreted in a more geometric context as a resolution of some sheaves of constant functions on generalized flag varieties. Čap, Slovák and Souček pushed further this approach, interpreting the generalized Verma modules used in the BGG resolution as jet bundles over the generalized flag varieties. This observation allowed them to generalized BGG sequences to manifolds with parabolic geometries in the sense of Cartan [83], i.e. manifolds "locally modeled" over a specific generalized flag variety. Here we try to generalize these ideas to foliated manifolds with transverse parabolic geometry, i.e. the leaf space is "locally modeled" over the generalized flag varieties. More precisely, since the BGG sequence can be constructed from the twisted De Rham complex on a tractor bundle, we show that in the context of transverse parabolic geometries, the twisted transverse De Rham complex shares the same properties and is modeled (i.e. on the symbolic level) on the Chevalley-Eilenberg complex with values in the representation used to build the tractor bundle. More precisely given a semi-simple Lie group G, a parabolic subgroup P and a manifold M with transverse (G, P) geometry (in a looser sense than in the literature), and a G-representation $\mathbb{E}$, we show that M is a foliated filtered manifold in the sense of chapter 4 and study a sequence of operators (the transversal twisted De Rham complex) $(\Omega^{0,\bullet}(M, E), \mathrm{d}^\nabla)$. Here $E \to M$ is the tractor bundle associated to the representation $\mathbb{E}$ of G and ∇ is the corresponding tractor connection. We then show that under a regularity condition on the geometry, the transversal osculating groupoid is identified to the trivial bundle with fiber the opposed nilpotent Lie group. We also show that under this condition the transversal complex becomes the Chevalley-Eilenberg cohomological complex at the transversal symbolic level. We can thus show that our sequence of operators satisfies a transversal graded Rockland condition. Using a Kostant type co-differential on the transverse forms, we define a sequence of BGG type operators on the corresponding homology bundles:

$$D_\bullet \colon \mathcal{H}_\bullet := (M \times H_\bullet(\mathfrak{p}_+, \mathbb{E})) \to \mathcal{H}_{\bullet+1}.$$

Using the result on the transverse de Rham sequence we prove that the transverse BGG sequence $(\mathcal{H}_\bullet, D_\bullet)$ is also transversally graded Rockland. Moreover if the transverse BGG sequence is a complex then so is the BGG sequence and the natural map from de Rham to BGG becomes an isomorphism in cohomology.

Chapter 2

Preliminaries

2.1 Groupoids and analysis

2.1.1 Lie groupoids and algebroids

Definition 2.1.1. A groupoid is the data of two sets M (set of objects) and G (the set of arrows) with structural maps $r, s\colon G \to M$ (range and source), $u\colon M \to G$ (unit), $i\colon G \to G$ (inversion) and $m\colon G^{(2)} := G\ {}_s\times_r G \to G$ (multiplication) satisfying the categorical axioms[1]:

- $s \circ u = r \circ u = \mathrm{Id}_M$.
- For all $(\gamma_1, \gamma_2) \in G^{(2)}$, $s(\gamma_1\gamma_2) = s(\gamma_2)$, $r(\gamma_1\gamma_2) = r(\gamma_1)$.
- For all $\gamma \in G$ we have $(\gamma, \gamma^{-1}), (\gamma^{-1}, \gamma) \in G^{(2)}$ and $\gamma\gamma^{-1} = u(r(\gamma)), \gamma^{-1}\gamma = u(s(\gamma))$.
- If $(\gamma_1, \gamma_2) \in G^{(2)}, (\gamma_2, \gamma_3) \in G^{(2)}$ then $\gamma_1(\gamma_2\gamma_3) = (\gamma_1\gamma_2)\gamma_3$.
- $\forall \gamma \in G, \gamma u(s(\gamma)) = \gamma = u(r(\gamma))\gamma$.

We have written $\gamma_1\gamma_2$ instead of $m(\gamma_1, \gamma_2)$ and γ^{-1} instead of $i(\gamma)$. It follows from the axioms that u is injective so from now on we will consider M as a subset of G and omit u. We write $G \rightrightarrows M$ for a groupoid G with space of objects M.

Definition 2.1.2. A Lie groupoid is a groupoid $G \rightrightarrows M$ with G and M being smooth manifolds, r, s are submersions (in particular $G^{(2)}$ is also a manifold), m is a submersion, u is an embedding and i a diffeomorphim.

[1] They just mean that M is a small category with arrows given by elements of G, m the composition of morphims, u gives the identity morphism. The existence of i means that every morphism is invertible.

Example 2.1.3. *Any manifold M can be seen as a Lie groupoid over itself with every structural map being trivial. On the opposite end of the spectrum a Lie group G can be seen as a groupoid over a singleton $G \rightrightarrows \{e\}$. Combining these two ideas: if $\pi \to G \to M$ is a group smooth field of Lie groups (i.e. there is a smooth submersion $m\colon G \times_\pi G \to G$ so that every $(\pi^{-1}(\{x\}), m_{|\pi^{-1}(\{x\})})$ is a Lie group) then it can be seen as a Lie groupoid with same range and source. For instance any vector bundle is a Lie groupoid over its base.*

Example 2.1.4. *Let M be a manifold, the pair groupoid is the groupoid $M \times M \rightrightarrows M$. The source and range are respectively the projection on the second and first arguments. The unit map is the diagonal embedding and the multiplication is given by the formula:*

$$\forall x, y, z \in M, (x, y) \cdot (y, z) = (x, z).$$

The inversion map is obtained by swapping the arguments: $(x, y)^{-1} = (y, x)$.

Example 2.1.5. *Let $G \curvearrowright M$ be a smooth group action. The action groupoid is the groupoid $G \ltimes M := G \times M \rightrightarrows M$. The structural maps are given as follows:*

- $s(g, x) = x, r(g, x) = gx$.
- $(h, gx) \cdot (g, x) = (hg, x)$.
- $(g, x)^{-1} = (g^{-1}, gx)$.
- $u(x) = (1_G, x)$ *where* 1_G *is the unit element of* G.

Example 2.1.6 (Connes' tangent groupoid)**.** *Let M be a smooth manifold. The tangent groupoid of M is $\mathbb{T}M \rightrightarrows M \times \mathbb{R}$ given algebraically as:*

$$\mathbb{T}M = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

The groupoid structure is given by the pair groupoid for $t \neq 0$ and the bundle group structure at 0. The smooth structure is obtained by a deformation to the normal cone (i.e. a tubular neighborhood embedding) by noticing that $TM = N_{\Delta_M}^{M \times M}$ where $\Delta_M \subset M \times M$ is the diagonal and N_V^W is the normal bundle of the embedding $V \to W$.

Given a Lie group one has its associated lie algebra. This functorial construction carries out for Lie groupoids to the Lie algebroid functor.

Definition 2.1.7. Let $G \rightrightarrows M$ be a Lie groupoid. The Lie algebroid of G is the vector bundle $\mathcal{A}G = N_M^G$ endowed with the following maps:

- the anchor map $\rho := \mathrm{d}r - \mathrm{d}s \colon \mathcal{A}G \to TM$
- a Lie bracket $[\cdot,\cdot]_{\mathcal{A}G}$ on the space of sections $\Gamma(\mathcal{A}G)$ (see the construction below)

The Lie bracket on the sections of $\mathcal{A}G$ is obtained as follows: any 1-parameter family of (local) bisections[2] $\gamma_t, t \in (-\epsilon, \epsilon)$ defines an element $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\gamma_t \in \Gamma_{loc}(\mathcal{A}G)$. Let $X, Y \in \Gamma_{loc}(\mathcal{A}G)$ and let $\gamma_t, g_t, t \in (-\epsilon; \epsilon)$ be local bisections of G such that the corresponding local sections of $\mathcal{A}G$ are respectively X and Y. Then the Lie bracket is given by:

$$[X,Y]_{\mathcal{A}G} = \frac{\partial}{\partial s}_{|s=0}\frac{\partial}{\partial t}_{t=0}\gamma_s g_t \gamma_s^{-1}.$$

Example 2.1.8.
- *The Lie algebroid of the trivial groupoid $M \rightrightarrows M$ is the 0 vector bundle over M.*
- *The Lie algebroid of a Lie group is its Lie algebra.*
- *The Lie algebroid of a smooth field of groups is the associated field of Lie algebras (which is also smooth by construction), the anchor is trivial.*
- *The Lie algebroid of the pair groupoid $M \times M \rightrightarrows M$ is the tangent bundle $TM \to M$ with the anchor equal to the identity map and the usual Lie bracket of vector fields.*
- *The Lie algebroid of an action groupoid $G \ltimes M$ is the action algebroid corresponding to the infinitesimal action $\mathfrak{g} \times M$. The differentiation of the action gives a map $\mathfrak{g} \to \mathfrak{X}(M)$ which is the anchor of the algebroid.*
- *The Lie algebroid of the tangent groupoid is the singular foliation given by the following module of sections[3]:*

$$\Gamma(\mathcal{A}\mathbb{T}M) = \{X \in \mathfrak{X}(M \times \mathbb{R}) \ / \ \partial_t X_{t|t=0} = 0\}.$$

A Lie algebroid on M is a more general object. It is a vector bundle $\mathcal{A} \to M$ with an anchor map $\rho \colon \mathcal{A} \to TM$ and a Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ on its space of sections satisfying the Leibniz rule:

$$\forall X, Y \in \Gamma(\mathcal{A}), \forall f \in \mathscr{C}^\infty(M), [X, fY]_{\mathcal{A}} = f[X,Y]_{\mathcal{A}} + \rho(X)(f)Y.$$

[2] A (local) bisection is a (local) section of the source map and its composition with the range map is a (local) diffeomorphism of M. They can be multiplied using the groupoid multiplication.

[3] A generalization of this idea has been used to define the tangent groupoid of a filtered manifold, see the next section below.

The problem of, given a Lie algebroid $\mathcal{A} \to M$, finding a Lie groupoid $G \rightrightarrows M$ such that $\mathcal{A}G = \mathcal{A}$ (we say that G integrates $\mathcal{A}$) is a very deep problem. It somehow corresponds to integrating Lie algebras of infinite dimension but that have a "geometric" origin (the Lie algebra is $(\Gamma(\mathcal{A}), [\cdot,\cdot]_{\mathcal{A}})$) and encompasses a lot of different problems. The most notable ones are the integration of foliations (i.e. subalgebroids of TM) and the integration of Poisson algebroids. The problem of integrability of a Lie algebroid is known as being impossible in general, the first counter-examples can be traced back to Almeida and Molino [1] and the general obstructions have been given by Crainic and Fernandes [25].

We end this section with a brief description of the holonomy groupoid of a foliation. A foliation is a subalgebroid F of TM. It is thus a subbundle of TM stable under the Lie bracket of vector fields. In this case the problem of integrability described above has a solution given by the holonomy groupoid. According to Frobenius theorem, each F_x corresponds to the tangent space of a submanifold called the leaf of a foliation passing through x. A submanifold $T \subset M$ is a transversal manifold if it means the leaves transversally i.e. $\forall x \in T, T_xT \oplus F_x = T_xM$, one can always construct such submanifolds locally. A holonomy is a diffeomorphism between local transversals to the foliation such that the image of a point is on the same leaf as the point.

Definition 2.1.9. The holonomy groupoid of a foliated manifold (M, F) is the groupoid $\mathrm{Hol}(F) \rightrightarrows M$ whose space of arrows is given by the germs of holonomies between the source and the range (in particular there can be morphims between two points iif the points are on the same leaf). The composition is the composition of germs of holonomies.

Proposition 2.1.10. *The holonomy groupoid is a Lie groupoid. It is source connected*[4] *and its orbits are the leaves of the foliation.*

Proof. Let $((\Omega_i, \phi_i))_{i\in I}$ be a foliated atlas. This means that the $\Omega_i, i \in I$ define an open covering of M, that $\phi_i\colon \Omega_i \to U_i \times T_i \subset \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}$ and the transition maps preserve this decomposition of $\mathbb{R}^p \times \mathbb{R}^q$, i.e. $\phi_{i,j}$ is of the form $(x, y) \mapsto (g_{i,j}(x, y), h_{i,j}(y))$. Define $W_{i,j,h} = U_i \times U_j \times T'_h$ where $h\colon T'_h \subset T_j \to T_h \subset T_i$ is a holonomy. We define a groupoid structure on $\sqcup_{i,j,h} W_{i,j,h}$ the following way:

- $s(u, v, t) = \phi_j^{-1}(v, t)$
- $r(u, v, t) = \phi_i^{-1}(u, h(t))$

[4]For each $x \in M$ the space $s^{-1}(x)$ is connected.

- $(u_1, v_1, t_1) \in W_{i,j,h}$ and $(u_2, v_2, t_2) \in W_{j,k,h'}$ are composable iif $v_1 = u_2$ and $t_1 = h'(t_2)$. In this case we have

$$(u_1, v_1, t_1) \cdot (u_2, v_2, t_2) = (u_1, v_2, t_2) \in W_{i,k,h \circ h'}.$$

Now define an equivalence relation on $\sqcup_{i,j,h} W_{i,j,h}$ the following way: we take holonomies $(i_1, j_1, h_1), (i_2, j_2, h_2)$ and a point $(x, y) \in (r, s)(W_{i_1,j_1,h_1}) \cap (r, s)(W_{i_2,j_2,h_2})$. Let $(u, t) = j_2(y)$, the holonomies h_1, h_2 define the same germ at y iif $h_1 \circ h_{j_1,j_2}$ and $h_{i_1,i_2} \circ h_2$ coincide on a neighborhood of t. This defines an equivalence relation $\sim$ on $\sqcup_{i,j,h} W_{i,j,h}$. The quotient $\widetilde{\mathrm{Hol}}(F) := {\sqcup_{i,j,h} W_{i,j,h}}/{\sim}$ is a Lie groupoid over M and integrates the algebroid F. The holonomy groupoid is its source-connected component. □

The holonomy groupoid can be constructed for more general foliations. A Stefan-Sussmann foliation is a locally finitely generated submodule of $\mathfrak{X}_c(M)$ closed under Lie brackets (in the regular case described above it corresponds to $\Gamma_c(F)$). In that regard the regular case corresponds to the locally free submodules. In this case there is still a notion of holonomy groupoid. First progress were made by Debord [29] who showed the integrability of any almost-injective algebroids (i.e. the anchor map is injective over an open dense subset of M). The general case was treated by Androulidakis and Skandalis [3].

2.1.2 The convolution algebra of a groupoid

In this section we consider a Lie groupoid $G \rightrightarrows M$ and construct a convolution product on $\mathscr{C}_c^\infty(G)$. From this algebra we can obtain the reduced and maximal groupoid C^* algebras of the groupoid. In the case of groups they are the usual reduced and maximal group C^*-algebras. In the framework on noncommutative geometry one should think of these algebras as the appropriate algebras of functions over the space of orbits of the groupoid (which has no good properties in general, e.g. not being Haussdorff). The construction can be done for any locally compact groupoid endowed with a Haar system (see [72]). Here we construct the convolution algebra with the help of half-densities [5], bypassing the need of a Haar system, but only suited to Lie groupoids. The advantage of this method is the simplification of the definitions for distributions below.

Let V be a real vector space and $\alpha \in \mathbb{R}$. The space of α-densities is the vector space:

$$\Omega^\alpha V := \{f \colon \Lambda^{\dim(V)} V \to \mathbb{R} \ / \ \forall x \in \Lambda^{\dim(V)} V, \forall \lambda \in \mathbb{R}, f(\lambda x) = |\lambda|^\alpha f(x)\}.$$

[5] The idea goes back to Connes for the holonomy groupoid of a foliation [20].

This definition extends directly to vector bundles to obtain a line bundle. The transition maps will be the absolute value of the jacobian determinant of the transition charts of the initial bundle, raised to the power $-\alpha$. In particular for $\alpha = 1$ the change of variable formula allows to define the integral of a 1-density on TM.

Let $G \rightrightarrows M$ be a Lie groupoid. The bundle of half-densities is

$$\Omega^{1/2} := \Omega^{1/2}(\ker(\mathrm{d}s)) \otimes \Omega^{1/2}(\ker(\mathrm{d}r)).$$

Instead of the function space $\mathscr{C}_c^\infty(G)$ we will consider the space of sections $\Gamma_c(G, \Omega^{1/2})$ which we will still denote by $\mathscr{C}_c^\infty(G, \Omega^{1/2})$ if no other bundle appears. Let $f, g \in \mathscr{C}_c^\infty(G, \Omega^{1/2})$, we define their convolution product with the following formula:

$$f * g(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2).$$

In other words, we first consider the function

$$f \otimes g \in \Gamma_c(G^2, \mathrm{pr}_1^* \Omega^{1/2} \otimes \mathrm{pr}_2^* \Omega^{1/2}).$$

We then restrict it to the submanifold of composable pairs $G^{(2)} \subset G^2$, we write $\iota\colon G^{(2)} \to G^2$ so that the restriction is the pullback

$$\iota^*(f \otimes g) \in \Gamma_c(G^{(2)}, \mathrm{pr}_1^* \Omega^{1/2} \otimes \mathrm{pr}_2^* \Omega^{1/2}).$$

Lemma 2.1.11. *On $G^{(2)}$ we have the isomorphim:*

$$\mathrm{pr}_1^* \Omega^{1/2} \otimes \mathrm{pr}_2^* \Omega^{1/2} \cong \Omega^1(\ker(\mathrm{d}m)) \otimes m^*(\Omega^{1/2}).$$

Proof. The map $G^{(2)} \to G \times_s G$ given by $(\gamma_1, \gamma_2) \mapsto (\gamma_1\gamma_2, \gamma_2)$ is an isomorphism and gives $\ker(\mathrm{d}m) \cong \mathrm{pr}_2^*(\ker(\mathrm{d}s))$ and also $m^*(\ker(\mathrm{d}s)) \cong \ker(\mathrm{d}\,\mathrm{pr}_2))$ but $\ker(\mathrm{d}\,\mathrm{pr}_2)) = \mathrm{pr}_1^*(\ker(\mathrm{d}s))$ so $m^*(\ker(\mathrm{d}s)) \cong \mathrm{pr}_1^*(\ker(\mathrm{d}s))$. We have the same result by using r instead of s reversing the roles of pr_1 and pr_2. We thus get:

$$pr_1^*(\ker(\mathrm{d}r) \oplus \ker(\mathrm{d}s)) \otimes \mathrm{pr}_2^*(\ker(\mathrm{d}r) \oplus \ker(\mathrm{d}s)) \cong m^* \ker(\mathrm{d}s) \oplus \ker(\mathrm{d}m) \oplus \ker(\mathrm{d}m) \oplus m^*(\ker(\mathrm{d}r)).$$

We then use the following properties of the density bundle construction:

$$\Omega^\alpha(V \oplus W) = \Omega^\alpha V \otimes \Omega^\alpha W$$
$$\Omega^\alpha V \otimes \Omega^\beta V = \Omega^{\alpha+\beta} V$$

and get the result. □

It follows from the lemma that we can integrate the one density part of $\iota^*(f \otimes g)$ along the m-fibers. Let us denote this map as m_*, we have

$$m_* \colon \Gamma_c(G^{(2)}, \mathrm{pr}_1^* \Omega^{1/2} \otimes \mathrm{pr}_2^* \Omega^{1/2}) \to \Gamma_c(G, m_*(m^*(\Omega^{1/2})) = \mathscr{C}_c^\infty(G, \Omega^{1/2}).$$

This let us write the convolution of half-densities as follows:

$$f * g = m_*(\iota^*(f \otimes g)). \tag{2.1}$$

This approach will allow a generalization to distributions on Lie groupoids in the next section following [60].

2.1.3 Convolution of distributions on a Lie groupoid

In this section we are interested in the convolution of distributions on Lie groupoids. Since pseudodifferential operators on a manifold arise as convolution operators by their Schwartz kernel, their composition reads as the convolution of their respective kernels. We thus need to understand the condition one needs to define such an operation. To extend the formula (2.1) to distributions we need to understand three kinds of operations on distributions:

- tensor product of distributions
- pullback of distributions
- push-forward of distributions by a submersion.

Example 2.1.12. *Let $M \rightrightarrows M$ be the trivial groupoid structure on a manifold. The convolution algebra is then the algebra of functions $\mathscr{C}_c^\infty(M)$ with the pointwise product.*

It is well known that there is no reasonable multiplication law on distributions. The example thus shows that there will be no general convolution of distributions and that one needs to add some conditions. These conditions will be on the wave front set. Let us recall the definition and basic properties of the wave front set of a distribution, they can be found in [48, 41].

Let $v \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution. Its Fourier transform $\hat{v}(\xi) := \langle v, e^{i\xi\cdot} \rangle$ is well defined and by the Paley-Wiener-Schwartz theorem, satisfies the estimates

$$|\hat{v}(\xi)| \leq C(1 + |\xi|)^M$$

for certain constants C, M. Moreover $v \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ iif

$$\forall N \in \mathbb{N}, \exists C_N, \forall \xi \in \mathbb{R}^n, |\hat{v}(\xi)| \leq C_N(1 + |\xi|)^{-N}$$

i.e. $\hat{v}$ has rapid decay. This property leads to the definition:

$$\Sigma(v) = \mathbb{R}^n \setminus \{\xi \in \mathbb{R}^n \ / \ \exists V \text{ conical neighborhood of } \xi, \hat{v}_{|V} \text{ has rapid decay } \}.$$

This property can be localized, indeed if $\phi \in \mathscr{C}^\infty(\mathbb{R}^n)$ then $\Sigma(\phi v) \subset \Sigma(v)$. We can thus localize the singularities around a given point $x \in \mathbb{R}^n$:

$$\Sigma_x(v) = \bigcap_{\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n), \phi(x) \neq 0} \Sigma(\phi v).$$

Moreover since we localize using compactly supported functions, this definition extends to any distribution in $\mathcal{D}'(\mathbb{R}^n)$.

Definition 2.1.13. The wave front set of a distribution $v \in \mathcal{D}'(\mathbb{R}^n)$ is the set

$$WF(v) = \sqcup_{x \in \text{ssing}(v)} \Sigma_x(v).$$

Here $\text{ssing}(v)$ denotes the singular support of the distribution.

It follows from the localization property and the Paley-Wiener-Schwartz theorem that $\Sigma_x(v) \neq \emptyset$ iif $x \in \text{ssing}(v)$, so we could also have made the union over every point of $\mathbb{R}^n$. The definition of the wave front set is compatible with composition by diffeomorphims and thus passes to distributions on manifolds. The wave front set then becomes a subset of the cotangent bundle of the manifold. The wave front set is a closed conic subset of the cotangent bundle. Its projection on the base manifold is the singular support. Its intersection with the fiber T_x^*X is the set $\Sigma_x(v)$.

Let $\Gamma \subset T^*X$ be a closed conic subset, we denote by $\mathcal{D}'_\Gamma(X), \mathcal{E}'_\Gamma(X)$ the sets of distributions and compactly supported distributions that have their wave front set contained in Γ.

Proposition 2.1.14. *Let X, Y be manifolds, $u \in \mathcal{D}'(X), v \in \mathcal{D}'(Y)$ then their tensor product is defined on $\mathscr{C}_c^\infty(X) \otimes \mathscr{C}_c^\infty(Y)$ as:*

$$\langle u \otimes v, f \otimes g \rangle := \langle u, f \rangle \langle v, g \rangle.$$

It extends to a distribution $u \otimes v \in \mathcal{D}'(X \times Y)$ and its wave front set satisfies:

$$WF(u \otimes v) \subset (WF(u) \cup \text{ssing}(u) \times \{0\}) \times (WF(v) \cup \text{ssing}(v) \times \{0\}).$$

Proposition 2.1.15. *Let $f \colon X \to Y$ be a smooth function. Denote by*

$$f^* \colon \mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(X)$$

the pullback by f. Let $\Gamma \subset T^*Y$ be a closed conic subset such that $\Gamma \cap N_f = \emptyset$ where $N_f \subset T^*Y$ is the conormal bundle to the graph of f[6]*, i.e.*

$$N_f = \{(y,\eta) \in T^*Y \ / \ \exists x \in X, f(x) = y \& {}^t\mathrm{d}_x f(\eta) = 0\}.$$

Then the pullback extends continuously to $f^* \colon \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'_{f^*\Gamma}(X)$ where:

$$f^*\Gamma := \{(x,\xi) \in T^*X \ / \ \exists \eta \in T^*_{f(x)}Y, (f(x),\eta) \in \Gamma \& {}^t\mathrm{d}_x f(\eta) = \xi\}.$$

Proposition 2.1.16. *Let X, Y be manifolds, $f \colon X, Y$ a smooth function. The push-forward by f is the function $f_* \colon \mathcal{E}'(X) \to \mathcal{D}'(Y)$ defined by:*

$$\forall u \in \mathcal{E}'(X), \forall \varphi \in \mathscr{C}^\infty_c(Y), \langle f_*(u), \varphi \rangle := \langle u, f^*\varphi \rangle.$$

Moreover if $\Gamma \subset T^*X$ is a closed conic subset then $f_*(\mathcal{E}'_\Gamma(X)) \subset \mathcal{D}'_{f_*\Gamma}(Y)$ where:

$$f_*\Gamma = \{(y,\eta) \in T^*Y \ / \ \exists x \in X, y = f(x) \& (x, {}^t\mathrm{d}_x f(\eta)) \in \Gamma \cup (X \times \{0\}\}.$$

We can now define the convolution of distributions on a Lie groupoid. Let $G \rightrightarrows M$ be a Lie groupoid. Denote by $\mathcal{D}'(G, \Omega^{1/2})$ and $\mathcal{E}'(G, \Omega^{1/2})$ the respective topological duals of $\Gamma_c(G, \Omega^{-1/2} \otimes \Omega_G)$ and $\Gamma(G, \Omega^{-1/2} \otimes \Omega_G)$ where $\Omega_G = \Omega^1(TG)$. This choice gives canonical embeddings:

$$\mathscr{C}^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{D}'(G, \Omega^{1/2}), \mathscr{C}^\infty_c(G, \Omega^{1/2}) \hookrightarrow \mathcal{E}'(G, \Omega^{1/2}).$$

Note that we have $\Omega^{1/2}_G \cong \Omega^{1/2}(\ker(\mathrm{d}r)) \otimes r^*(\Omega^{1/2}_M)$, the same goes for s and thus

$$\Omega^{-1/2} \otimes \Omega_G \cong r^*(\Omega^{1/2}_M) \otimes s^*(\Omega^{1/2}_M).$$

From the previous propositions, we see that the only problem in extending the convolution product to distributions on a groupoid is the pullback by ι. We need to prove find conditions on distributions $u, v \in \mathcal{E}'(G, \Omega^{1/2})$ so that $WF(u \otimes v) \cap N_\iota = \emptyset$. A general condition is given in [60] involving the product on the cotangent groupoid of G in the sense of Coste-Dazord-Weinstein [24].

Definition 2.1.17. Let $\pi \colon M \to B$ be a smooth submersion. A distribution $u \in \mathcal{E}'(M)$ is transverse to π if:

$$\forall f \in \Gamma_c(M, \Omega^1_M), \pi_*(fu) \in \Gamma_c(B, \Omega^1_B).$$

We denote by $\mathcal{E}'_\pi(M)$ and $\mathcal{D}'_\pi(M)$ the corresponding subspaces of distributions.

[6]More precisely it is the projection on T^*Y of the conormal bundle to the graph.

It is shown in [60] that these distributions correspond to smooth families of distributions on the fibers of the fibration. Using the wave front set we have a large class of examples of distributions.

Proposition 2.1.18. *Let $\pi\colon M \to B$ be a smooth submersion. If $\Gamma \subset T^*M$ is a closed conic subset such that $\Gamma \cap \ker(\mathrm{d}\pi)^\perp = \emptyset$ then $\mathcal{D}'_\Gamma(M) \subset \mathcal{D}'_\pi(M)$.*

Proof. Let $u \in \mathcal{D}'_\Gamma(M)$ and $f \in \mathscr{C}^\infty(M, \Omega^1_M)$. Use the formula for the wave front set of a push forward:

$$WF(\pi_*(fu)) \subset \pi_*(WF(fu)) \subset \pi_*(WF(u)) \subset \pi_*\Gamma.$$

The condition $\Gamma \cap \ker(\mathrm{d}\pi)^\perp = \emptyset$ then gives $\pi_*\Gamma = \emptyset$. □

The converse statement is not true in general. We now give one of the results of [60] on convolution of distributions. It does not use conditions of wave front sets directly. We have chosen to use them however to stress the problems in defining the convolution of distributions. Moreover this approach allows a more general statement which is more suitable for Fourier integral operators. In particular if one is only interested in G-pseudodifferential operators then one could only look at the convolution on $\mathcal{D}'_{\mathcal{A}^*G}(G)$.

Theorem 2.1.19 (Lescure, Manchon, Vassout [60])**.** *Let $G \rightrightarrows M$, $\pi = r$ or s, the convolution product defined by formula (2.1) extends to $\mathcal{E}'_\pi(G, \Omega^{1/2})$ and makes it a associative algebra. Moreover the Dirac distribution δ on the unit space defined as*

$$\forall \varphi \in \Gamma(G, \Omega^{-1/2} \otimes \Omega_G), \langle \delta, \varphi \rangle := \int_M \varphi$$

is the unit of the algebra $(\mathcal{E}'_\pi(G, \Omega^{1/2}), *)$. If we denote by

$$\mathcal{E}'_{r,s}(G, \Omega^{1/2}) = \mathcal{E}'_r(G, \Omega^{1/2}) \cap \mathcal{E}'_s(G, \Omega^{1/2})$$

the space of distributions both r and s-fibered, then it is also a unital algebra and it is stable under the involution:

$$u \mapsto u^* := \overline{i^*(u)}.$$

2.2 Calculus on filtered manifolds

2.2.1 Geometry of filtered manifolds

Osculating groupoid

Definition 2.2.1. A filtered manifold is a smooth manifold M with the additional data of subbundles $H^1 \subset \cdots \subset \cdots H^r$ with the condition on the

Lie brackets:

$$\forall i, j, \left[\Gamma(H^i), \Gamma(H^j)\right] \subset \Gamma(H^{i+j})$$

with the convention that $H^k = TM$ whenever $k \geq r$. The data of $(H^i)_{i\geq 1}$ is called a Lie filtration on M. We will sometimes refer to the number r as the step of the filtration.

Example 2.2.2.

- *Any manifold is trivially filtered by $H^1 = TM$.*
- *A contact manifold is a filtered manifold with H^1 the contact hyperplane distribution and $H^2 = TM$.*
- *Let $(M, \mathcal{F})$ be a foliated manifold, $H^1 = T\mathcal{F}$ and $H^2 = TM$ give M the structure of a filtered manifold.*
- *Let G be a real semisimple Lie group and $P \subset G$ a parabolic subgroup. Then the homogeneous space ${}^{G}\!/_{P}$ is naturally endowed with a Lie filtration. This extends more generally to manifolds carrying a parabolic geometry in the sense of Cartan. This example is the motivation for chapter 5 and will be discussed in more details there.*

In the examples above the contact manifolds and foliated manifolds give two examples of step one filtrations. They are however very different in their geometry as can be witnessed by the Frobenius theorem (in that regard they stand as opposites on the spectrum of integrability). We need a tool to capture this difference: the osculating groupoid.

Let us define

$$\mathfrak{t}_H M = H^1 \oplus \cdots \oplus {}^{H^r}\!/_{H^{r-1}}$$

as a vector bundle over M. For $1 \leq i, j \leq n$ and $X \in \Gamma(H^i), Y \in \Gamma(H^j)$ we have $[X, Y] \in \Gamma(H^{i+j})$ and the value of $[X, Y] \mod \Gamma(H^{i+j-1})$ only depends on $X \mod \Gamma(H^{i-1})$ and $Y \mod \Gamma(H^{j-1})$. The Lie bracket of vector fields thus induces a map $[\cdot, \cdot] \colon \Lambda^2(\Gamma(\mathfrak{t}_H M)) \to \mathbb{R}$ i.e. a Lie algebroid structure on $\mathfrak{t}_H M$ (with trivial anchor).

Lemma 2.2.3. *The map $[\cdot, \cdot] \colon \Lambda^2(\Gamma(\mathfrak{t}_H M)) \to \mathbb{R}$ is tensorial and thus comes from a Lie algebra structure on each fiber of $\mathfrak{t}_H M$ (smooth when the base point varies).*

Proof. Let $1 \leq i, j \leq r$, $X \in \Gamma(H^i), Y \in \Gamma(H^j)$ and $f \in \mathscr{C}^\infty(M)$. Because $i \geq 1$ we have $Y \equiv 0 \mod \Gamma(H^{i+j-1})$, thus:

$$[fX, Y] = -X(f)Y + f[X, Y] \equiv f[X, Y] \mod \Gamma(H^{i+j-1}).$$

The same goes for the other variable. □

Corollary 2.2.3.1. *$\mathfrak{t}_H M$ is a bundle of graded Lie algebras[7] over M.* [8]

Proof. The Lie algebra has already been proved. The graded part is by construction of $\mathfrak{t}_H M$ and its Lie bracket. □

Definition 2.2.4. Given a filtered manifold (M, H) we associate to it its osculating groupoid $T_H M$ which is the smooth family of (connected, simply connected) Lie groups integrating $\mathfrak{t}_H M$. The total space of this bundle can be considered as the same as $\mathfrak{t}_H M$ and the product constructed through the Baker-Campbell-Haussdorff formula.

Example 2.2.5.

- *On a manifold with trivial filtered structure we have $T_H M = TM$ seen as a bundle of abelian groups.*
- *If M is a contact manifold with contact form θ then $T_H M$ is a (locally trivial) bundle of Heisenberg groups. The fiber at x is canonically identified with $\mathrm{Heis}(\ker(\theta_x), \mathrm{d}_x\theta)$ where $(\ker(\theta_x), \mathrm{d}_x\theta)$ is seen as a symplectic vector space. The local triviality is given by any choice of Darboux coordinates.*
- *Let $(M, \mathcal{F})$ be a foliated manifold. Then $T_H M = T\mathcal{F} \oplus {}^{TM}\!/_{T\mathcal{F}}$ seen as a bundle of abelian groups.*

All the examples above might give the (wrong) idea that $T_H M$ is always locally trivial, it is however not always the case.

Example 2.2.6. *Let $M = \mathbb{R}^3$ and $X = \partial_x + y^2\partial_z, Y = \partial_y, Z = \partial_z$. Consider $H =< X, Y >$ as a rank 2 subbundle of TM. One has $[X, Y] = -2yZ$ and thus: $T_{H,(x,y,z)}M$ is the Heisenberg group of dimension 3 for $y \neq 0$ and the abelian group $\mathbb{R}^3$ otherwise.*

Proposition 2.2.7. *The assignment $(M, H) \mapsto T_H M$ is a functor from the category of filtered manifolds with maps preserving the filtration to the category of Lie groupoids.*

Proof. Let $f\colon (M, H) \to (M', H')$ be a map preserving the filtration i.e. $\forall i \geq 1, \mathrm{d}f(H^i) \subset H'^i$. This implies that $\mathrm{d}f$ faktors as maps

$$\mathrm{d}f\colon {}^{H^i}\!/_{H^{i-1}} \to {}^{H'^i}\!/_{H'^{i-1}},$$

[7]The Lie algebra structure is not necessarily locally trivial

[8]A graded Lie algebra is a Lie algebra of the form $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \geq 1$, with the convention $\mathfrak{g}_k = \{0\}$ for $k > r$.

resulting in a linear map $f_*\colon \mathfrak{t}_H M \to \mathfrak{t}_{H'} M'$. The push-forward of vector fields is compatible with their Lie brackets and thus f_* is a Lie algebroid morphim, i.e. it preserves the Lie algebra structure on each fiber. Since the exponential map is a diffeomorphism between the Lie algebroid and the corresponding Lie groupoid (the fibers are nilpotent) we obtain a morphism $f_*\colon T_H M \to T_{H'} M'$[9]. □

A generalization of this setup can be the following: if $G \rightrightarrows M$ is a Lie groupoid with filtered algebroid. It means that there is a filtration

$$\mathcal{A}^1 G \subset \cdots \subset \mathcal{A}^r G = \mathcal{A}G,$$

with the condition

$$\forall i, j \geq 1, \left[\Gamma(\mathcal{A}^i G), \Gamma(\mathcal{A}^j G)\right]_{\mathcal{A}G} \subset \Gamma(\mathcal{A}^{i+j} G).$$

We can adapt the construction to the osculating groupoid by replacing TM with $\mathcal{A}G$ and obtain a groupoid $\mathcal{A}_H G$ replacing the algebroid (which is again a smooth family of graded Lie groups). Most of the results stated in this thesis can be adapted to these groupoids with filtered algebroid but for the sake of simplicity we will often state them for filtered manifolds (which correspond to pair groupoids $M \times M \rightrightarrows M$).

Example 2.2.8. *Let $H^1 \subset \cdots \subset H^r \subset TM$ satisfying the same conditions than a Lie filtration but we replace the condition $H^r = TM$ by only requiring H^r to be an integrable subbundle. We can thus consider H^r as a foliation and the filtration as a filtration of the Lie algebroid of the corresponding holonomy groupoid. This corresponds to a holonomy invariant filtered structure on the leaves of the foliation.*

Remark 2.2.9. Another generalization of this setup has been proposed and studied in [2]. It corresponds to replacing the subbundles of TM by finitely generated $\mathscr{C}^\infty(M)$-submodules of the Lie algebra of vector fields $\mathfrak{X}(M)$ generalizing the modules of sections $\Gamma(H^i)$. This context is by nature more singular and the objects involved are more subtle: for instance the dimension of the osculating group might not even be constant. Although I am confident that the results contained in this thesis will generalize to this singular case I have not worked on it yet and will only consider filtered manifolds as described above.

[9] One can directly use the fact that $T_H M$ is source connected and source simply connected

The osculating groupoid are equipped with inhomogeneous dilations. At the level of the Lie algebroid they are defined as

$$\forall \lambda > 0, \forall i \geq 1, \forall X \in \Gamma\left({H^i}/{H^{i-1}}\right), \mathrm{d}\delta_\lambda X = \lambda^i X.$$

These $\mathrm{d}\delta_\lambda$ define Lie algebroid automorphisms and thus lift to automorphisms δ_λ of the osculating groupoid $T_H M$. The morphisms f_* constructed in the last proposition are equivariant w.r.t these $\mathbb{R}_+^*$-actions. We can extend the definition of these morphisms to $\lambda \in \mathbb{R}$, they will only be automorphims for $\lambda \neq 0$.

Although there is no "local triviality" in the filtered calculus as the structure of the osculating groupoid might change abruptly we can still understand the representations of the osculating groupoid through the ones of its fibers. More generally we will sometimes consider bundles[10] of graded Lie groups. A bundle of graded Lie groups is a Lie groupoid with same range and source maps $G \to M$ such that the fibers $G_x, x \in M$ are simply connected Lie groups with graded nilpotent Lie algebras[11]. The filtration on the Lie algebra of a graded Lie group gives rise to a one parameter subgroup of automorphisms $\delta_\lambda \colon G \to G$ given at the level of the Lie algebra by $\forall i, \mathrm{d}\delta_{\lambda|\mathfrak{g}_i} = \lambda^i \operatorname{Id}$, this extends fiberwise in the case of bundles of graded Lie groups.

Example 2.2.10. *$T_H M \to M$ is a bundle of graded Lie groups. If G is a graded Lie group then $G \to \{e\}$ is a bundle of graded Lie groups.*

Proposition 2.2.11. *Let $G \to M$ be a bundle of graded Lie groups. The algebra $C^*(G)$ is a continuous field of C^*-algebras over M, the fiber at $x \in M$ is equal to the group C^*-algebra $C^*(G_x)$.*

Proof. This directly follows from the fact that G is a smooth (hence continuous) family of group(oid)s over M in the sense of [56]. □

Remark 2.2.12. Every bundle of graded Lie groups is amenable (the fibers are nilpotent Lie groups) hence there is no difference between the reduced and maximal groupoid C^*-algebras.

Deformation groupoid

In [21], Connes defined the tangent groupoid and noticed its importance in index theory. Algebraically it is defined as the disjoint union:

$$\mathbb{T}M = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}$$

[10] We will use the word by simplicity but as before the group structure might not be locally trivial

[11] These groups and algebras are also called "Carnot" in the literature

with $M \times M \rightrightarrows M$ seen as the pair groupoid and $TM \rightrightarrows M$ as a bundle of abelian group. The two are glued together by a deformation to the normal cone, noticing that $TM \cong N_{\Delta_M}^{M\times M}$.

In the last 10 years, the link between the tangent groupoid and pseudodifferential calculus has been better understood. Following the work of Debord and Skandalis [30] and later Van Erp and Yuncken [81], the tangent groupoid has been used as a tool to define pseudodifferential calculus by gluing a pseudodifferential operator to its principal symbol. We will explain the construction of Van Erp and Yuncken in the next sections (in the general context of calculus on filtered manifolds). The work of Debord and Skandalis is the root of the work appearing in chapter 3.

In order to deal with the construction of Van Erp and Yuncken in the case of filtered manifolds we need an analog of Connes tangent groupoid in this context. Let (M, H) be a filtered manifold and $T_H M$ its osculating groupoid. The tangent groupoid should glue the pair groupoid with the osculating group bundle instead of the tangent bundle, i.e. be algebraically of the form:

$$\mathbb{T}_H M = M \times M \times \mathbb{R}^* \sqcup T_H M \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

The easiest,although abstract, way of defining $\mathbb{T}_H M$ is to first define its algebroid $\mathbb{t}_H M$ through its module of sections:

$$\Gamma(\mathbb{t}_H M) = \{X \in \mathfrak{X}(M \times \mathbb{R}) \;/\; \forall k \geq 1, \partial_t^k X_{|t=0} \in \Gamma(H^k)\}.$$

Proposition 2.2.13. $\Gamma(\mathbb{t}_H M)$ *is a module of section of a vector bundle over* $M \times \mathbb{R}$*. Its fiber at* $t \in \mathbb{R}$ *is isomorphic to* TM *for* $t \neq 0$ *and* $\mathfrak{t}_H M$ *for* $t = 0$*.*

Proof. Let $\psi \colon \mathfrak{t}_H M \to TM$ be a vector bundle isomorphim such that the restriction of ψ to each ${H^i}/{H^{i-1}}$ is a right inverse to the natural quotient map[12]. The map $\Psi \colon \Gamma(\mathfrak{t}_H M \times \mathbb{R}) \to \Gamma(\mathbb{t}_H M)$ defined as $\Psi(X)(x,t) = \psi(\delta_t(X(x)))$ is well defined and is an isomorphism of $\mathscr{C}^\infty(M \times \mathbb{R})$-modules. In particular $\mathbb{t}_H M$ is the module of section of a vector bundle over $M \times \mathbb{R}$. The isomorphism on the fibers is given directly at $t \neq 0$ by the obvious evaluation map $X \mapsto X_t \in \mathfrak{X}(TM)$. The isomorphism at $t = 0$ is given by $X \mapsto \left(\frac{1}{k!}\partial_t^k X_{|t=0} \mod H^{k-1}\right)_{k\geq 1}$. □

The algebroid structure on $\mathbb{t}_H M$ is then given by the following maps:

- The anchor is $\rho(X) = X$ (in particular it is trivial at $t = 0$)

[12] ψ is called a splitting of $\mathfrak{t}_H M$ in the terminology of [80]

- The Lie bracket is given by $[X,Y](x,t) = [X_t, Y_t](x)$.

The fact that the Lie bracket restricts to $\Gamma(\mathbb{t}_H M)$ follows from the fact that

$$\partial_t^k [X,Y]_{|t=0} = \sum_{j=0}^{k} \left[\partial_t^j X_{|t=0}, \partial_t^{k-j} Y_{|t=0}\right]$$

and the Lie bracket conditions on the Lie filtration.

The evaluation maps at $t \neq 0$ and the twisted evaluation at $t = 0$ are compatible with the bracket and the anchor. The algebroid $\mathbb{t}_H M$ is thus identified, algebraically, to the disjoint union:

$$\mathbb{t}_H M = TM \times \mathbb{R}^* \sqcup \mathfrak{t}_H M \times \{0\}.$$

This algebroid is almost injective (the anchor map is injective over the dense subset $M \times \mathbb{R}^*$). It then follows from a theorem of Debord [29] that $\mathbb{t}_H M$ is integrable by a Lie groupoid $\mathbb{T}_H M$. This groupoid is algebraically equal to the disjoint union:

$$\mathbb{T}_H M = M \times M \times \mathbb{R}^* \sqcup T_H M \times \{0\}.$$

Several authors have given more concrete descriptions of the smooth structure of $\mathbb{T}_H M$, see [18, 80, 63]. We briefly recall the approach of Van Erp and Yuncken in [80] as it will be useful in chapter 3.

Let ∇ be a connection on $\mathfrak{t}_H M$ equivariant for the $\mathbb{R}_+^*$-action and choose a splitting $\psi\colon \mathfrak{t}_H M \to TM$. The isomorphism $\Psi\colon \mathfrak{t}_H M \times \mathbb{R} \to \mathbb{t}_H M$ allows to push forward the connection to a Lie algebroid connection written as a family $(\nabla^t)_{t\in\mathbb{R}}$ with $\nabla^0 = \nabla$ and $\nabla^t = \psi \circ \nabla \circ \psi^{-1}$ for $t \neq 0$. With these connections one has an exponential map $\exp^{\nabla,\psi}\colon \mathbb{t}_H M \to \mathbb{T}_H M$. It is shown in [80] that this groupoid exponential is injective on a neighborhood of $t = 0$ which gives the smooth structure on $\mathbb{T}_H M$. They also show that this smooth structure is independent of the choices of ∇ and ψ.

More precisely a neighborhood of the zero-section $\mathcal{U} \subset \mathfrak{t}_H M$ is called a domain of injectivity if $\exp^{\psi\circ\nabla\circ\psi^{-1}} \circ \psi\colon \mathcal{U} \to M \times M$ is well-defined and injective[13]. From a domain of injectivity they construct global exponential charts. Define $\mathbb{U} = \{(x,\xi,t) \in \mathfrak{t}_H M \times \mathbb{R} \;/\; (x, \delta_t(\xi)) \in \mathcal{U}\}$, then $\exp^{\nabla,\psi} \circ \Psi$ is well defined and injective on $\mathbb{U}$. The image of $\mathbb{U}$ by the exponential map is then the open subset

$$\exp^{\psi\circ\nabla\circ\psi^{-1}}(\psi(\mathcal{U})) \times \mathbb{R}^* \sqcup T_H M \times \{0\} \subset \mathbb{T}_H M.$$

[13]Here $\exp^{\psi\circ\nabla\circ\psi^{-1}}$ denotes the groupoid exponential on the groupoid $M \times M$, it is defined as $\exp^{\psi\circ\nabla\circ\psi^{-1}}(x,v) = (x, \exp_x^{\psi\circ\nabla\circ\psi^{-1}}(v))$, the second exponential being the usual exponential map of a connection.

Moreover we have the explicit formula:

$$\begin{aligned}\exp^{\nabla,\psi} \circ \Psi \colon (x, \xi, t) &\mapsto \left(\exp^{\psi\circ\nabla\circ\psi^{-1}}(x, \psi(\delta_t(\xi)), t\right) \\ (x, \xi, 0) &\mapsto (x, \xi, 0)\end{aligned}$$

Remark 2.2.14. In [63] the functorial properties of $\mathbb{T}_H M$ appear more clearly and are analogous to the functoriality of the usual tangent groupoid construction (replacing the differential of a function by the functoriality of the osculating groupoid at $t = 0$). This allows to extend directly the construction of the tangent groupoid in the filtered setting to an adiabatic groupoid of groupoids with filtered algebroids. If $G \rightrightarrows M$ is a Lie groupoid with filtered algebroid, recall there is an osculating groupoid $\mathcal{A}_H G$ replacing the algebroid. Then one can construct in the same fashion as $\mathbb{T}_H M$ an adiabatic extension

$$\mathbb{G}_H^{ad} = G \times \mathbb{R}^* \sqcup \mathcal{A}_H G \times \{0\}$$

with a natural Lie groupoid structure. In particular this adiabatic extension allows to extend directly the calculus for filtered manifolds that will be explained in the next sections to a groupoid setting.

We can extend the action by the inhomogeneous dilations on $T_H M$ to the so called zoom action on $\mathbb{T}_H M$. For $\lambda > 0$ define $\alpha_\lambda \in \mathrm{Aut}(\mathbb{T}_H M)$ as follows:

$$\begin{aligned}\alpha_\lambda(x, y, t) &= (x, y, \lambda^{-1} t) \\ \alpha_\lambda(x, \xi, 0) &= (x, \delta_\lambda(\xi), 0).\end{aligned}$$

In a global exponential chart the action becomes

$$\tilde{\alpha_\lambda}(x, \xi, t) = (x, \delta_\lambda(\xi), \lambda^{-1} t).$$

This action is smooth thus the corresponding zoom action on $\mathbb{T}_H M$ is also smooth.

Example 2.2.15. *Let G be a graded Lie group seen as a filtered manifold. Then the tangent groupoid has the structure of an action groupoid:*

$$\mathbb{T}_H G \cong (G \times \mathbb{R}) \rtimes G$$

with $(h, t) \cdot g = (h\delta_t(g), t)$. Under this identification the zoom action becomes $\lambda \cdot (h, t, g) = (h, \dfrac{t}{\lambda}, \delta_\lambda(g))$.

Homogeneous quasi-norms

On a graded Lie group bundle a natural way to "measure" the size of vectors is a homogeneous quasi-norm. It replaces norms on a vector bundle in a way that is compatible with the inhomogeneous dilations.

Definition 2.2.16. Let $G \to M$ be a graded Lie group bundle. A homogeneous quasi-norm on G is a continuous function $|\cdot|: G \to \mathbb{R}_+$ such that:

- $|g| = 0$ iif $g = 1_x$ for some $x \in M$
- $|\cdot|$ is homogeneous of degree 1, i.e. $\forall g \in G, \forall \lambda > 0, |\delta_\lambda(g)| = \lambda|g|$

Example 2.2.17. *Let $X_1, \cdots, X_m$ be a local basis of the fiber of $\mathfrak{g}$ such that for every i, $\mathrm{d}\delta_\lambda(X_i) = \lambda^{n_i} X_i$ and denote by $n = \sum_{i=1}^{m} n_i$ the homogeneous dimension. For $t_1, \cdots, t_n \in \mathbb{R}$ set:*

$$\left|\exp(\sum_{i=1}^{m} t_i X_i)\right| := \left(\sum_{i=1}^{m} t_i^{2n/n_i}\right)^{1/2n}.$$

It is then clear that it satisfies the requirements of a homogeneous quasi-norm on the open subset of M over which the X_i are defined. This construction can be carried over to the whole bundle by using a partition of unity on M. One can also do a construction that resembles the L^1-norm with:

$$\left|\exp(\sum_{i=1}^{m} t_i X_i)\right| := \sum_{i=1}^{m} |t_i|^{1/n_i}.$$

These two constructions show that one can have homogeneous quasi-norms that are smooth on $G \setminus \{1_M\}$.

Since the group bundles we use are exponential we can see the homogeneous quasi-norms on $\mathfrak{g}$ instead of G. From that we can transfer it to the dual Lie algebra bundle $\mathfrak{g}^* \to M$ with the formula:

$$\forall x \in M, \forall \eta \in \mathfrak{g}_x^*, |\eta| := \sup_{\xi \in \mathfrak{g}_x \setminus \{0\}} \frac{|\eta(\xi)|}{|\xi|}.$$

By construction the quasi-norm on $\mathfrak{g}^*$ is still homogeneous of degree 1 with respect to the dual dilations ${}^t\mathrm{d}\delta_\lambda$. We can now push it forward to the quotient $\mathfrak{g}^*/G$ for the co-adjoint action with the formula:

$$|\operatorname{Ad}^*(G)\eta| := \inf_{g \in G} |\operatorname{Ad}^*(g)\eta|.$$

The co-adjoint action is compatible with the dilations:

$$ {}^t\mathrm{d}\delta_\lambda(\mathrm{Ad}^*(g)\eta) = \mathrm{Ad}^*(\delta_\lambda(g))^t\mathrm{d}\delta_\lambda(\eta), $$

thus the resulting map $|\cdot|\colon {\mathfrak{g}^*}/{G} \to \mathbb{R}$ is equivariant and vanishes only at the points of the form $[0_{\mathfrak{g}_x^*}], x \in M$. We will see in the following parts 2.3 that the quotient space ${\mathfrak{g}^*}/{G}$ is homoemorphic to the dual space $\hat{G}$ of irreducible representations with the hull-kernel topology. In chapter 3 we will be interested in the space ${\hat{G}\setminus\{1\}}/{\mathbb{R}_+^*}$ and the homogeneous quasi-norm will then allow us to realize it as the closed subset

$$ \{\pi \in \hat{G}, |\pi| = 1\} \subset \hat{G}. $$

2.2.2 Symbolic calculus

Symbols in the filtered calculus

Let us recall the definition of symbols in the context of filtered calculus. In the following, $\pi\colon G \to M$ denotes a bundle of graded Lie groups. In this section the groupoids $T_H M$ and $T_{H/H^0}M$ will be used as examples. The construction of $T_H M$ has already been detailed in the previous parts. The construction of $T_{H/H^0}M$ is postponed until chapter 4. In the meantime it should be thought as a analog of $T_H M$ replacing TM but for the transverse bundle ${TM}/{H^0}$ to a foliation $H^0 \subset TM$.

Definition 2.2.18. A symbol of order $m \in \mathbb{C}$ in G is a distribution $u \in \mathcal{D}'(G, \Omega^{1/2})$ satisfying:

- u is properly supported, i.e. $\pi\colon \mathrm{supp}(u) \to M$ is a proper map
- u is transversal to π (in the sense of [3])
- $\forall \lambda \in \mathbb{R}_+^*, \delta_{\lambda *}u - \lambda^m u \in \mathscr{C}_p^\infty(G, \Omega^{1/2})$[14]

$S_p^m(G)$ denotes the set of these distributions and $S_p^*(G) = \bigcup_{m\in\mathbb{Z}} S_p^m(G)$. The subspace of compactly supported symbols will be denoted by $S_c^m(G)$. We will write $S^m(G)$ when the statements apply for both $S_p^m(G)$ and $S_c^m(G)$.

The second condition allows us to "disintegrate" u along the fibers to obtain a C^∞ family of distributions $u_x \in \mathcal{D}'(G_x, \Omega^{1/2})$ (see [60] for a more precise statement and more details on transversality of distributions). The

[14]This condition will be called quasi-homogeneity in the remaining parts of the thesis.

first condition then implies that the support of each u_x is a compact subset of the fiber G_x for every $x \in M$, i.e. $u_x \in \mathcal{E}'(G_x, \Omega^{1/2})$. The third condition implies that the singularities of u are located on the zero section $M \subset G$ and also gives the asymptotic behavior of u near M.
The symbols used here being quasi-homogeneous, they will correspond to principal symbols of H-pseudodifferential operators up to $\mathscr{C}_p^\infty(G, \Omega^{1/2})$ half-densities. For compactly supported symbols, the quasi-homogeneity condition can equivalently be stated $\mod \mathscr{C}_c^\infty(G)$).
Note that here the push-forward of distributions is defined by duality with the pullback of half-densities. Let $\lambda > 0$ and $f \in \mathscr{C}_c^\infty(G, \Omega^{1/2})$ we have:

$$\delta_\lambda^* f = \lambda^{n/2} f \circ \delta_\lambda,$$

where n denotes the homogeneous dimension of G, i.e. $n = \sum_{i \geq 1} i \operatorname{rk}(\mathfrak{g}_i)$. A quick computation shows that, similarly:

$$\delta_{\lambda *} f = \lambda^{n/2} f \circ \delta_{\lambda^{-1}} = \lambda^n \delta_{\lambda^{-1}}^* f.$$

These morphisms define actions of $\mathbb{R}_+^*$ on $\mathscr{C}^\infty(G, \Omega^{1/2}), \mathscr{C}_c^\infty(G, \Omega^{1/2})$ and their respective duals, they are compatible with the convolution product for compactly supported functions and distributions.

Remark 2.2.19. As we use properly supported distributions we do the same for functions: $\mathscr{C}_p^\infty(\cdot)$ denotes the space of properly supported functions on a groupoid (if G is a groupoid, $X \subset G$ is proper if $s_{|X}$ and $r_{|X}$ are proper maps). With these notations we see that $\mathscr{C}_p^\infty(G, \Omega^{1/2})$ embeds into the space of properly supported fibered distributions on G (an arbitrary Lie groupoid). Although we will mainly use compactly supported symbols because of their analytic properties (see 2.2.28 below) we introduced properly supported distributions as they naturally arise when considering deformation groupoids in the framework of [81].

Remark 2.2.20. We will also use symbols acting on vector bundles. Let E, F be vector bundles over M. We write $S^0(G; E, F)$ for the space of distributions with values in $\operatorname{End}(E, F)$. This means that for every $x \in M$, $u_x \in \mathcal{D}'(G_x, \Omega^{1/2}) \otimes E_x^* \otimes F_x$. We also write $S^0(G; E) = S^0(G; E, E)$. In order to keep lighter notations the general results for symbols in the filtered calculus will be stated without vector bundles (note that we need $E = F$ in order to use the convolution product below).

We recall the useful results on filtered calculus, for more details and proofs we refer to [81, 64] and their references:

Lemma 2.2.21. *Let $u \in S^{m_1}(G), v \in S^{m_2}(G)$ then $u * v \in S^{m_1+m_2}(G)$ and $u^* \in S^{m_1}(G)$. This result can be extended to $m_1 = -\infty$ or $m_2 = -\infty$ with $S^{-\infty}_{c/p}(G) = \mathscr{C}^\infty_{c/p}(G, \Omega^{1/2})$.*

Let $u \in S^*(G)$, for each $x \in M$ we get a convolution operator

$$\begin{array}{rcccc} \mathrm{Op}(u_x) & : & \mathscr{C}^\infty_c(G_x, \Omega^{1/2}) & \longrightarrow & \mathscr{C}^\infty_c(G_x, \Omega^{1/2}) \\ & & f & \longmapsto & u_x * f \end{array}.$$

The transversality condition then allows us to "glue" these operators to obtain

$$\mathrm{Op}(u) \colon \mathscr{C}^\infty_c(G, \Omega^{1/2}) \to \mathscr{C}^\infty_c(G, \Omega^{1/2}),$$

with $\mathrm{Op}(u)(f)_{|\pi^{-1}(x)} = \mathrm{Op}(u_x)(f_{|\pi^{-1}(x)})$.

Lemma 2.2.22. *Let $u, v \in S^*(G)$ and $f, g \in \mathscr{C}^\infty_c(G, \Omega^{1/2})$ then*

- $\mathrm{Op}(u)(f * g) = \mathrm{Op}(u)(f) * g$
- $\mathrm{Op}(u * v) = \mathrm{Op}(u) \circ \mathrm{Op}(v)$

Lemma 2.2.23. *Let G_1, G_2 be bundles of graded Lie groups over M and $\phi \colon G_1 \to G_2$ a submersive homomorphism of bundle groups which is grade preserving. ϕ induces a map $\phi_* \colon S^*(G_1) \to S^*(G_2)$ that preserves :*

- *The order*
- *The sum of elements of the same degree*
- *The adjoint*
- *The convolution product*

Proof. To show that ϕ_* indeed maps $S^*(G_1)$ to $S^*(G_2)$, we need to show that it preserves transversality and quasi-homogeneity. For transversality it is obvious since ϕ is a bundle map. Moreover since ϕ is a morphism between graded Lie groups bundles then it is equivariant with respect to the respective $\mathbb{R}^*_+$-actions on G_1 and G_2. The conservation of the quasi-homogeneity thus reduces to the fact that $\phi_*(\mathscr{C}^\infty_p(G_1, \Omega^{1/2})) \subset \mathscr{C}^\infty_p(G_2, \Omega^{1/2})$. The proof is then very similar to the one of lemmas 4.2.9 and 4.2.10 in chapter 4 so we refer to these further proofs.
The sum, convolution product and adjoint are then clearly preserved by ϕ_*. □

Example 2.2.24. *It is well known that for any Lie group (or groupoids), elements of the universal enveloping algebra correspond to left-invariant differential operators on the group(oid), see [68]. We can do the same for the filtered calculus. Let* $G \to M$ *be a graded Lie groups bundle,* $\mathfrak{g} \to M$ *the corresponding Lie algebra bundle. The universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ *is fibered over* M *with fiber at* $x \in M$ *equal to* $\mathcal{U}(\mathfrak{g}_x)$*. By the universal property of the enveloping algebra, the morphisms* $\mathrm{d}\delta_\lambda \colon \mathfrak{g} \to \mathfrak{g}$ *extend to* $\delta_{\lambda *} \colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$*. This family of morphisms allow to modify the notion of order on* $\mathcal{U}(\mathfrak{g})$ *in the obvious way. Elements of* $\Gamma(\mathfrak{g}_i)$ *give elements of order* i *and if* $X_{i_1}, \cdots, X_{i_m} \in \Gamma(\mathfrak{g})$ *are respectively in* $\Gamma(\mathfrak{g}_{i_1}), \cdots, \Gamma(\mathfrak{g}_{i_m})$ *then their product* $\prod_{k=1}^{m} X_{i_k} \in \Gamma(\mathcal{U}(\mathfrak{g}))$ *is an element of order* $\sum_{k=1}^{m} i_k$*. If we now consider elements of* $\Gamma(\mathcal{U}(\mathfrak{g}))$ *as left-invariant differential operators on* G *then an element of* $\Gamma(\mathcal{U}(\mathfrak{g}))$ *of (modified) order* m *gives an element of* $S^m_p(G)$*.*

We now state the Rockland condition and its consequences on the symbolic calculus. Rockland condition replaces the usual ellipticity condition. Moreover, when the symbols yield operators on M (i.e. for $G = T_H M$, see [81]) the operators whose symbols satisfy Rockland's condition are hypoelliptic and admit parametrices (in the filtered calculus). This condition was first introduced by Rockland in [73] for differential operators on Heisenberg groups. It was then extended to manifolds with osculating Lie group of rank at most two by Helffer and Nourrigat in [43] and Lie groups with dilations in [19]. See also the recent advances in [2] in a very broad setting generalizing the results of Helffer and Nourrigat.

Definition 2.2.25. A symbol $u \in S^*(G)$ satisfies the Rockland condition at $x \in M$ if there exist a compact set $K \subset \widehat{G_x}$ on the dual space of irreducible representations such that for all $\pi \notin K$, $\pi(\mathrm{Op}(u))$ and $\pi(\mathrm{Op}(u^*))$ are injective on the space of smooth vectors. The symbol satisfies the Rockland condition if it satisfies it at every point $x \in M$ (u is then also said to be Rockland).

Example 2.2.26. *If* $H^1 = TM$*,* $T_H M$ *and* $T_{H/H^0} M$ *are abelian group bundles hence*
$\widehat{T_H M}_x = T^*_x M$ *and* $\widehat{T_{H/H^0} M}_x = (H^0)^\perp$ *so that Rockland condition for symbols on* TM *and* $TM/_{H^0}$ *corresponds respectively to the classical notions of ellipticity and transverse ellipticity.*

As in the classical case, Rockland condition corresponds exactly to the existence of parametrices. It was proved in [19] for trivial bundles and this result was used in [27] for the proof in the general case.

Theorem 2.2.27 ([27])**.** *Let* $u \in S^m_p(G)$ *then there is an equivalence :*

1 u satisfies the Rockland condition

2 there exists $v \in S_p^{-m}(G)$ such that $u * v - 1, v * u - 1 \in \mathscr{C}_p^{\infty}(G)$

A proof can be found in [27], using arguments originating in [19] (which corresponds to the case of trivial bundles).

Theorem 2.2.28 ([27])**.** *Let $u \in S_c^m(G)$ then*

- *If $m \leq 0$ then $\mathrm{Op}(u)$ extends to an element of the multiplier algebra $\mathcal{M}(C^*(G))$*
- *If $m < 0$ then $\mathrm{Op}(u)$ extends to an element of $C^*(G)$*
- *If $m > 0$, M is compact and u satisfies the Rockland condition then $\mathrm{Op}(u)$ extends to an unbounded regular operator $\overline{\mathrm{Op}(u)}$ on $C^*(G)$[15] and $\overline{\mathrm{Op}(u)}^* = \overline{\mathrm{Op}(u^*)}$*

Once again see [27] for a detailed proof.

Definition 2.2.29. A symbol $u \in S^*(T_H M)$ is Rockland (i.e "elliptic" in the filtered calculus) if it satisfies the Rockland condition. It is transversally Rockland if its image in $S^0(T_{H/H^0} M)$ satisfies the Rockland condition.

Theorem 2.2.28 allows to extend the algebra of symbols. Let

$$\bar{S}_0^0(G) \subset \mathcal{M}(C^*(G))$$

be the C^*-closure of $S_c^0(G)$ (we identify the algebra of order 0 symbols and its image by $\overline{\mathrm{Op}}$). The algebra $\bar{S}_0^0(G)$ corresponds to the algebra of symbols vanishing at infinity. If M is non-compact, such symbols cannot satisfy the Rockland condition and have a convolution inverse in $\bar{S}_0^0(G)$. We thus need to extend the algebra of symbols to the one of symbols "bounded" at infinity. More precisely, following [44], let

$$C_M^*(G) = \{T \in \mathcal{M}(C^*(G)) \ / \ \forall f \in \mathscr{C}_0(M), \ fT, Tf \in C^*(G)\},$$
$$\bar{S}^0(G) = \{T \in \mathcal{M}(C^*(G) \ / \ \forall f \in \mathscr{C}_0(M), \ fT, Tf \in \bar{S}_0^0(G)\}.$$

We will also need the corresponding algebras of principal symbols:

$$\Sigma_c^m(G) = {}^{S_c^m(G)}\!\big/_{\mathscr{C}_c^{\infty}(G,\Omega^{1/2})},$$
$$\Sigma_0^0(G) = {}^{\bar{S}_0^0(G)}\!\big/_{C^*(G)},$$
$$\Sigma^0(G) = {}^{\bar{S}^0(G)}\!\big/_{C_M^*(G)}.$$

[15] Viewed as a C^*-module over itself.

Note that the Rockland condition only depends on the class of a symbol modulo $\mathscr{C}_p^\infty(G)$ and is thus an invertibility criterion in $\Sigma^0(G)$.

Example 2.2.30. *If $G = G_0 \times M$ is a trivial bundle then*

$$C^*(G) = \mathscr{C}_0(M) \otimes C^*(G), C^*_M(G) = \mathscr{C}_b(M) \otimes C^*(G).$$

Every symbol in $S_p^0(G)$ that extends continuously to a multiplier of $C^*(G)$ (e.g. multiplication by functions in $\mathscr{C}_p^\infty(G)$) defines an element of $\bar{S}^0(G)$.

Let $k > 0$ and $D \in \Gamma(\mathcal{U}_k(\mathfrak{g}))$ be a Rockland differential operator. Assume D to be self-adjoint and non-negative and M to be a compact manifold. The third point of 2.2.28 allows to construct $\overline{\mathrm{Op}(D)}^{\mathrm{i}t} \in M(C^*(G))$ through functional calculus for $t \in \mathbb{R}$. One would hope to have some actual symbol $D^{\mathrm{i}t} \in S_p^{\mathrm{i}kt}(G)$ that would extend to the multiplier obtained through functional calculus. This is possible (even for the pseudodifferential operators, see 2.2.37 below) thanks to estimates on the heat kernels in the filtered calculus obtained in [26].

Theorem 2.2.31 ([26])**.** *Let $G \to M$ be a graded Lie groups bundle on a compact base. Let $D \in \Gamma(\mathcal{U}_k(\mathfrak{g}))$ be Rockland, essentially self-adjoint and non-negative with $k > 0$ an even number. There is a family of symbols $D^{\mathrm{i}t} \in S_p^{\mathrm{i}kt}(G)$ such that $\mathrm{Op}(D^{\mathrm{i}t})$ extends to the multiplier $\overline{\mathrm{Op}(D)}^{\mathrm{i}t}$ obtained from $\overline{\mathrm{Op}(D)}$ through functional calculus. These symbols are also compatible with the product: $D^{\mathrm{i}t_1}D^{\mathrm{i}t_2} = D^{\mathrm{i}(t_1+t_2)}$.*

We end this section by an analog of 2.2.11 for the algebra of principal symbols.

Proposition 2.2.32. *Let $G \to M$ be a bundle of graded Lie groups. The algebra of principal symbols $\Sigma(G)$ is a continuous field of C^*-algebras over M with fiber at $x \in M$ equal to $\Sigma(G_x)$.*

Proof. It is clear from the fact that the distributions are fibered over M that $\Sigma(G)$ is a $\mathscr{C}_0(M)$-algebra and the fiber at $x \in M$ is equal to $\Sigma(G_x)$. The definition of the norm from 2.2.28 makes $x \mapsto \|\sigma_x\|$ continuous for $\sigma \in \Sigma(G)$ by 2.2.11. $\square$

Remark 2.2.33. If $G = G_0 \times M$ is a trivial bundle then

$$\Sigma(G) = \Sigma(G_0) \otimes \mathscr{C}_0(M).$$

Equivariance of distributions

In chapter 4 we will consider the action of the holonomy groupoid of a foliation on transverse symbols. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, $E \to M$ a vector bundle and $G \to M$ a bundle of graded Lie groups. Assume $\mathcal{G}$ acts on G and E, i.e. for all $\gamma \in \mathcal{G}$, $\gamma\colon G_{s(\gamma)} \to G_{r(\gamma)}$ is a group homomorphism and $\gamma\colon E_{s(\gamma)} \to E_{r(\gamma)}$ is a linear map.

We want to understand the action of $\mathcal{G}$ on the spaces of symbols $S^m(G;E)$. Let $\sigma \in S^m(G;E)$, consider the distributions $s^*(\sigma), r^*(\sigma)$ defined respectively by $s^*(\sigma)_\gamma = \sigma_{s(\gamma)}, r^*(\sigma)_\gamma = \sigma_{r(\gamma)}$. We have

$$s^*(\sigma) \in S^m(s^*G; s^*E), r^*(\sigma) \in S^m(r^*G; r^*E).$$

The action of $\mathcal{G}$ on the space of symbols is defined by the map

$$\alpha\colon S^m(s^*G; s^*E) \to S^m(r^*G; r^*E),$$

defined by $\alpha(u)_\gamma = \gamma \circ u_\gamma \circ \gamma^{-1}$. A symbol $\sigma \in S^m(G;E)$ is thus equivariant if $\alpha(s^*(\sigma)) = r^*(\sigma)$, meaning that for every $\gamma \in \mathcal{G}$,

$$\gamma \circ \sigma_{s(\gamma)} = \sigma_{r(\gamma)} \circ \gamma.$$

It is equivariant modulo smoothing operators if

$$\alpha(s^*(\sigma)) - r^*(\sigma) \in \Gamma_c(r^*T_HM; \pi^*r^*\operatorname{End}(E) \otimes \Omega^{1/2}).$$

Equivariance can also be considered for $\operatorname{Op}(\sigma)$. The actions on $S^m(G;E)$ and $\Gamma_c(G; \pi^*E \otimes \Omega^{1/2})$ are compatible i.e. Op is equivariant. This means that a symbol σ is $\mathcal{G}$-equivariant if and only if the associated convolution operator $\operatorname{Op}(\sigma)\colon \Gamma_c(G; \pi^*E \otimes \Omega^{1/2}) \to \Gamma(G; \pi^*E \otimes \Omega^{1/2})$ is $\mathcal{G}$-equivariant.

For $m = 0$ the action of $\mathcal{G}$ on symbols extends continuously to $\bar{S}^0_0(G;E)$ and preserves the ideal $C^*(G) \otimes_M \operatorname{End}(E)$. It thus defines an action on $\Sigma^0_0(G;E)$. The action also extends to the other algebras of "bounded" symbols. We will then say that a symbol is equivariant modulo compact operators if its class of principal symbol is equivariant. This means for a symbol $\sigma \in \bar{S}^0_0(G)$ that $\alpha(s^*(\sigma)) - r^*(\sigma) \in r^*C^*(G) \otimes_M \operatorname{End}(E)$. In particular an operator invariant modulo smoothing operator is invariant modulo compact operators.

2.2.3 Pseudodifferential operators

Definition 2.2.34. A pseudodifferential operator in the filtered calculus on M of order $m \in \mathbb{C}$ is a compactly supported distribution $P \in \mathcal{E}'(M \times M, \Omega^{1/2})$ such that there exists $\mathbb{P} \in \mathcal{D}'(\mathbb{T}_HM, \Omega^{1/2})$ with the following properties:

- $\mathbb{P}$ is properly supported
- $\mathbb{P}_1 = P$ where $\mathbb{P}_t = \mathrm{ev}_{t*}\mathbb{P}$ and ev_t is the evaluation map on the fiber at time $t \in \mathbb{R}$ on $\mathbb{T}_H M$.
- $\mathbb{P}$ is quasi-homogeneous w.r.t the zoom action i.e.

$$\forall \lambda > 0, \alpha_{\lambda *}\mathbb{P} - \lambda^m \mathbb{P} \in \mathscr{C}_p^\infty(\mathbb{T}_H M, \Omega^{1/2}).$$

We denote by $\Psi_H^m(M)$ the set of such distributions and $\mathbb{\Psi}_H^m(M)$ the set of their extensions $\mathbb{P}$ to $\mathbb{T}_H M$. We use $m = -\infty$ for the space of regularizing operators $\mathscr{C}_p^\infty(M \times M, \Omega^{1/2})$.

From this setup one can derive the usual properties of a pseudodifferential calculus, namely the algebra structure, the existence of a symbol map, ellipticity criterion, parametrices... The symbol of an operator $P \in \Psi_H^m(M)$ is here obtained by extending it to some quasi-homogeneous $\mathbb{P}$ and considering $\mathbb{P}_0 \in S^m(T_H M)$. This will depend on the choice of $\mathbb{P}$ but the class $[\mathbb{P}_0] \in \Sigma_c^m(T_H M)$[16] only depends on P.We get the exact sequence :

$$0 \longrightarrow \Psi_H^{m-1}(M) \longrightarrow \Psi_H^m(M) \longrightarrow \Sigma_c^m(T_H M) \longrightarrow 0\,.$$

Theorem 2.2.35 (Dave, Haller [27])**.** *Let $P \in \Psi_H^m(M)$ then:*

- *If $\Re(m) \leq 0$, then P extends to a bounded operator on $L^2(M)$.*
- *If $\Re(m) < 0$, then P extends to a compact operator on $L^2(M)$.*

Proof. Follows from the estimates on the full symbol of the operator obtained in [81], corollary 45. □

This result can be refined to the whole extension $\mathbb{P}$ of a pseudodifferential operator P, it will be discussed at the end of chapter 4. Let us denote by $\Psi_H^*(M)$ the closure of $\Psi_H^0(M)$ in $\mathcal{B}(L^2(M))$. The previous exact sequence for $m = 0$ extends to the following one:

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi_H^*(M) \longrightarrow \Sigma(T_H M) \longrightarrow 0\,.$$

The usual ellipticity condition is here replaced by the Rockland condition on symbols, we will say that a pseudodifferential operator satisfies the Rockland condition if its symbol does.

[16] We can take an extension of P such that every $\mathbb{P}_t$ has its support included in some compact set $K \subset M \times M$. This forces $\mathbb{P}_0$ to have compact support.

Theorem 2.2.36 (Van Erp, Yuncken [81])**.** *An operator $P \in \Psi_H^m(M)$ is Rockland if and only if it admits a parametrix, i.e. there exists $Q \in \Psi_H^{-m}(M)$ such that $PQ - 1, QP - 1 \in \Psi_H^{-\infty}(M)$.*

The last thing we will need is the existence of complex powers for some positive operators:

Theorem 2.2.37 (Dave, Haller [26])**.** *Let M be a compact filtered manifold. Let P be a differential operator of positive even order r which satisfies the Rockland condition and is positive. Then the complex powers $P^z, z \in \mathbb{C}$, obtained through functional calculus are pseudodifferential operators of respective order rz. Moreover these operators form a holomorphic family of pseudodifferential operators.*

Remark 2.2.38. The definition of the pseudodifferential calculus extends to vector bundles valued operators. If E, F are vector bundles over M then one can construct a class of operators $\Psi_H^m(M; E, F)$ by replacing $\mathcal{E}'(\mathbb{T}_H M, \Omega^{1/2})$ with $\mathcal{E}'(\mathbb{T}_H M, \hom(s^*(E \times \mathbb{R}), r^*(F \times \mathbb{R})) \otimes \Omega^{1/2})$. We can also consider graduations on the vector bundles so that they will also be equipped with inhomogeneous dilations but we will come back to it in chapter 5.

2.2.4 The Schwartz algebra

Of a nilpotent Lie group

We recall briefly what the Schwartz algebra of a (simply connected) nilpotent Lie group is. This relies on the fact that these groups are exponential so we can push-forward the Schwartz algebra of their Lie algebra by the exponential map. Let $G \to M$ be a graded Lie group bundle and $\mathfrak{g} \to M$ its Lie algebroid. If we see $\mathfrak{g} \to M$ as a vector bundle it has an intrisically defined Schwartz algebra. Indeed if V is a vector space then choosing a basis gives an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$ and we can pullback the Schwartz algebra $\mathscr{S}(\mathbb{R}^n)$ to V. The resulting algebra of functions on V does not depend on the choice of basis so that the algebra has an intrinsic (i.e. functorial) meaning[17]. By functoriality this construction extends directly to vector bundles. We can now use the diffeomorphism $\exp\colon \mathfrak{g} \xrightarrow{\sim} G$ to define the algebra $\mathscr{S}(G)$. The algebra structure is given by the point-wise product which is not really relevant on a group. To consider the groupoid convolution we replace the Schwartz functions by Schwartz half-densities: $\mathscr{S}(G, \Omega^{1/2})$. Note that in this group case, the groupoid source and range maps are the same. Therefore

[17] A different basis would give the same algebra but would change the family of seminorms to a different (but still equivalent) one.

the half-densities bundle corresponds to the one-densities on the group. It can be shown that the groupoid convolution of G defines a continuous map:

$$\mathscr{S}(G,\Omega^{1/2})\times\mathscr{S}(G,\Omega^{1/2})\to\mathscr{S}(G,\Omega^{1/2}).$$

Therefore when we refer in the next sections and chapters to the Schwartz algebra of G it will be for the algebra structure given by the convolution product. Moreover the algebra is stable under the adjoint map

$$f\mapsto f^*=\overline{\iota^* f},$$

where $\iota\colon G\to G$ is the group inversion. This makes $\mathscr{S}(G,\Omega^{1/2})$ an involutive algebra. We then easily get the inclusions:

$$\mathscr{C}_c^\infty(G,\Omega^{1/2})\subset\mathscr{S}(G,\Omega^{1/2})\subset C^*(G).$$

Another approach is considered in [38]. In this approach the authors consider a homogeneous quasi-norm on G and define the Schwartz class by replacing the usual euclidean norm on $\mathbb{R}^n$ by the quasi-norm on G and differential operators with polynomial coefficients on $\mathbb{R}^n$ by elements of the universal envelopping algebra $\mathcal{U}(\mathfrak{g})$. It turns out that the euclidean norm and the homogeneous quasi-norm give equivalent growth conditions and we can recover the same types of operators in both ways. Therefore both approaches result in the same algebra with the same topology, the families of semi-norms are slightly different but equivalent nonetheless.

Of the tangent groupoid

We now define the Schwartz algebra of the tangent groupoid. Our approach is analogous to the one of Carillo-Rouse in the unfiltered case [15]. A similar approach has been used by Ewert in [35] but she used the coordinates of Choi and Ponge [18] while we will use the ones of Van Erp and Yuncken [80]. Both approaches yield the same algebra. The idea is to define the algebra on exponential charts and show that these algebra can be glued on the whole tangent groupoid. This Schwartz algebra will essentially contain two kinds of functions: the smooth functions with compact support on the open subgroupoid $\mathbb{T}_H M_{|\mathbb{R}^*}$ and Schwartz type functions on exponential charts (in particular their restriction on $T_H M$ will be Schwartz). In order to describe these last functions, we fix ∇ a graded connection on $\mathfrak{t}_H M$, compatible with the dilations and a splitting $\psi\colon \mathfrak{t}_H M\to TM$. Recall that this induces a vector bundle isomorphism $\Psi\colon \mathfrak{t}_H M\times\mathbb{R}\to\mathfrak{t}_H M$. We fix $\mathcal{U}\subset\mathfrak{t}_H M$ a domain of injectivity which gives us an open subset $\mathbb{U}\subset\mathfrak{t}_H M\times\mathbb{R}$ such that $\exp^{\nabla,\psi}\circ\Psi\colon\mathbb{U}\to\mathbb{T}_H M$ is a diffeomorphism onto its image $\mathbb{V}$. We have

$\mathbb{V} = (T_H M \times \{0\}) \sqcup (\exp^{\psi\circ\nabla\circ\psi^{-1}}(\mathcal{U}) \times \mathbb{R}^*)$ so $\mathbb{V}$ is both a neighborhood of the unit section of $\mathbb{T}_H M$ and of the zero-fiber $T_H M$. We thus have that $\mathbb{T}_H M = \mathbb{V} \bigcup (M \times M \times \mathbb{R}^*)$. We define the Schwartz algebra on $\mathbb{V}$ by defining one on $\mathbb{U}$ and pushing it forward with the exponential map.

Definition 2.2.39. The algebra $\mathscr{S}(\mathbb{V})$ is defined as $(\exp^{\nabla,\psi} \circ \Psi)_*(\mathscr{S}(\mathbb{U}))$. A function $f \in \mathscr{C}^\infty(\mathbb{U})$ is in $\mathscr{S}(\mathbb{U})$ if there is a compact set $K \subset \mathcal{U}$ and $T > 0$ such that if $(x, \delta_t\xi, t) \notin K \times [-T, T]$ then $f(x, \xi, t) = 0$ and also f is Schwartz as a function on the bundle of nilpotent Lie algebras $\mathfrak{t}_H M \times \mathbb{R} \to M \times \mathbb{R}$. A function $f \in \mathscr{C}^\infty(\mathbb{T}_H M)$ is a Schwartz type function if there exists an exponential chart $\exp^{\nabla,\psi} \circ \Psi \colon \mathbb{U} \to \mathbb{V}$ such that

$$f \in \mathscr{S}(V) + \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M)).$$

We denote by $\mathscr{S}(\mathbb{T}_H M)$ the space of such functions.

Remember that $\delta_0 = 0$ so in particular the whole $\mathfrak{t}_H M$ seen as the zero fiber of $\mathfrak{t}_H M \times \mathbb{R}$ can be included in the support of a function $f \in \mathscr{S}(\mathbb{U})$. For $t \neq 0$ however the support of $f_t = f(\cdot, \cdot, t)$ is contained in $\delta_t^{-1}(K)$ which is compact. In particular a smooth function on $\mathbb{U}$ is Schwartz if and only if it has the aforementioned support condition and is Schwartz at $t = 0$ (asking it on $\mathfrak{t}_H M \times \mathbb{R}$ is redundant).

Proposition 2.2.40. *Let $f = (f_t)_{t\in\mathbb{R}} \in \mathscr{S}(\mathbb{T}_H M)$ then:*

- *$f_0 \in \mathscr{S}(T_H M)$*
- *for every $t \neq 0$, $f_t \in \mathscr{C}_c^\infty(M \times M)$ moreover the support of each f_t is contained in a compact set of $M \times M$ independent of t*
- *the function $t \mapsto f_t$ has rapid decay when t goes to $\pm\infty$.*

Proof. These properties follow from the definition if the function is in the subalgebra $\mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M))$. If $f \in \mathscr{S}(\mathbb{V})$ where $\mathbb{V}$ is the range of an exponential chart then it follows directly from the definition of $\mathscr{S}(\mathbb{U})$ and the explicit computation of the exponential map $\exp^{\psi\circ\nabla\circ\psi^{-1}} \circ \Psi$. For the first point we have $\Psi_{|t=0} = \mathrm{Id}_{\mathfrak{t}_H M}$ so we recover the definition of the Schwartz class on a graded Lie group bundle. □

Theorem 2.2.41. *The definition of $\mathscr{S}(\mathbb{T}_H M)$ does not depend on the choice of connection, splitting and exponential chart. That means that for any connection ∇, splitting ψ, domain of injectivity $\mathcal{U}$ and corresponding exponential chart $\mathbb{U} \xrightarrow{\sim} \mathbb{V} \subset \mathbb{T}_H M$ then:*

$$\mathscr{S}(\mathbb{T}_H M) = \mathscr{S}(\mathbb{V}) + \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M)).$$

Proof. If $\mathbb{V}' \subset \mathbb{V}$ we can use a function φ compactly supported in $\mathbb{V}$ with value 1 near the zero fiber so that if $f \in \mathscr{S}(\mathbb{V})$ then $f = \varphi f + (1-\varphi)f$. Since $(1-\varphi)f \in \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M))$ we can just show that $\varphi f \in \mathscr{S}(\mathbb{V}')$. Therefore we can restrict ourselves to the case where $\mathbb{V}$ and $\mathbb{V}'$ are defined from the same domain of injectivity $\mathcal{U} \subset \mathfrak{t}_H M$ and show that if $\Psi, \Psi' \colon \mathfrak{t}_H M \times \mathbb{R} \xrightarrow{\sim} \mathfrak{t}_H M$ are the respective isomorphims used to construct the exponential charts then:

$$(\Psi^{-1} \circ \Psi')^*(\mathscr{S}(\mathbb{U})) = \mathscr{S}(\mathbb{U}').$$

Notice that $\Psi^{-1} \circ \Psi' = (\delta_t^{-1} \circ \psi^{-1} \circ \psi' \circ \delta_t)_{t \in \mathbb{R}}$, the value at $t = 0$ is Id. It is thus a vector bundle isomorphism of $\mathfrak{t}_H M \times \mathbb{R}$ and therefore preserves the Schwartz class. We now need to show that the condition on the support is preserved under the transformation given by $\Psi^{-1} \circ \Psi'$. For $K \subset \mathcal{U}$ and $T > 0$, we define:

$$K_T := \{(x, \xi, t) \in \mathfrak{t}_H M \times \mathbb{R}, t \in [-T; T] \& (x, \delta_t(\xi)) \in K\}.$$

By definition a function in $\mathscr{S}(\mathbb{U})$ has a support contained in a set of the form K_T for some $T > 0$ and $K \subset \mathcal{U}$ compact. The computation of $\Psi^{-1} \circ \Psi'$ shows directly that:

$$\Psi^{-1} \circ \Psi'(K_T) = (\Psi^{-1} \circ \Psi'(K))_T,$$

hence the condition on the support is preserved by the transformation $\Psi^{-1} \circ \Psi'$. □

In order to deal with convolution we go to Schwartz class for half-densities: $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$. They are the sections of the half-density bundle for which in one (hence any) zoom-invariant trivialization of the half-density bundle[18] they become Schwartz functions as defined before. To see that this definition is not ambiguous, let us consider two different zoom-invariant trivializations. They differ by a multiplication by a positive smooth function. By zoom invariance this function is zoom-invariant. In particular it is bounded and thus preserves the growth conditions at infinity used to define the Schwartz algebra.

Theorem 2.2.42. *The groupoid convolution and adjoint extend to the space* $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ *making it a* $*$*-algebra. Moreover we have the inclusions:*

$$\mathscr{C}_c^\infty(\mathbb{T}_H M, \Omega^{1/2}) \subset \mathscr{S}(\mathbb{T}_H M, \Omega^{1/2}) \subset C^*(\mathbb{T}_H M).$$

[18] This means that taking the pullback by α_λ scales the measure by a factor λ^{-d_H} where d_H is the homogeneous dimension of M.

Proof. From what we have written thus far we can easily see that the convolution of two functions in the Schwartz class is well defined fiberwise. It thus remains to show that the result of the convolution is still in the Schwartz class. Let us fix a splitting ψ, an equivariant connection ∇, consider a domain of injectivity $\mathcal{U} \subset \mathfrak{t}_H M$ and let $\mathbb{U} \subset \mathfrak{t}_H M \times \mathbb{R}$ the corresponding open subset and $\exp^{\nabla,\psi} \colon \mathbb{U} \xrightarrow{\sim} \mathbb{V} \subset \mathbb{T}_H M$ the resulting exponential chart. Since the convolution will force us to consider products in $\mathbb{T}_H M$ we choose an open subset $\mathcal{U}' \subset \mathcal{U}$ such that $m_{\mathbb{T}_H M}(\mathbb{V}' \; {}_s\!\times_r \mathbb{V}') \subset \mathbb{V}$. Let $f, g \in \mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$, we decompose them as $f = f_1 + f_2, g = g_1 + g_2$ with $f_1, g_1 \in \mathscr{S}(\mathbb{V}', \Omega^{1/2})$ and $f_2, g_2 \in \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$. We have:

$$f * g = f_1 * g_1 + f_1 * g_2 + f_2 * g_1 + f_2 * g_2,$$

and we easily see that $f_1 * g_2, f_2 * g_1, f_2 * g_2 \in \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2})$. We now show that $f_1 * g_1 \in \mathscr{S}(\mathbb{V}, \Omega^{1/2})$. At any non-zero fiber the restrictions f_1 and g_1 are compactly supported and so is the restriction of $f_1 * g_1$. The same goes for the zero fiber since the Schwartz class on a nilpotent Lie group is preserved under convolution. Thus the only remaining property to verify is the support condition for $f_1 * g_1$. Let $K, C \subset \mathcal{U}'$ be compact subsets and $T > 0$ such that we have the inclusions $\operatorname{supp}(f_1) \subset \exp^{\nabla,\psi}(K_T), \operatorname{supp}(g_1) \subset \exp^{\nabla,\psi}(C_T)$. Define $K \widetilde{\oplus} C \subset \mathfrak{t}_H M$ as:

$$\begin{aligned} exp^{\psi\circ\nabla\circ\psi^{-1}}(\psi(K\widetilde{\oplus}C)) = \Big\{ \Big(& \exp^{\psi\circ\nabla\circ\psi^{-1}}_{\exp_x^{\psi\circ\nabla\circ\psi^{-1}}(\psi(\xi))}(\psi(\eta)), x \Big), \\ & (x,\xi) \in K \ \& \ (\exp_x^{\psi\circ\nabla\circ\psi^{-1}}(\psi(\xi)), \eta) \in C \Big\}. \end{aligned}$$

This set is constructed so that $\exp^{\nabla,\psi}((K\widetilde{\oplus}C)_T)$ would contain all possible products (in $\mathbb{T}_H M$) of elements in $\operatorname{supp}(f_1)$ and $\operatorname{supp}(g_1)$ hence it would contain $\operatorname{supp}(f_1 * g_1)$. By construction the set $\exp^{\psi\circ\nabla\circ\psi^{-1}}(\psi(K\widetilde{\oplus}C))$ is contained in $\exp^{\nabla,\psi}(\mathbb{U})$ so that $K\widetilde{\oplus}C$ is well defined, compact and hence $f_1 * g_1 \in \mathscr{S}(\mathbb{V})$.

Now the fact that $\mathscr{C}_c^\infty(\mathbb{T}_H M, \Omega^{1/2}) \subset \mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ is rather obvious. Moreover since we also have $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2}) \subset L^1(\mathbb{T}_H M, \Omega^{1/2})$ then every continuous representation of $\mathbb{T}_H M$ extends continuously to $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ and thus $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2}) \subset C^*(\mathbb{T}_H M)$. □

Corollary 2.2.42.1. *The subalgebra* $\mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$ *is an ideal of* $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$.

2.3 Representations of nilpotent Lie groups

The representation theory of nilpotent Lie groups is well understood through Kirillov's homeomorphism between the unitary dual of a nilpotent Lie group with its hull-kernel topology and the set of its coadjoint orbits with the quotient topology obtained from the euclidean topology on the dual Lie algebra. In the case of graded Lie groups we can stratify the space of coadjoint orbits following the works of Puckanzky and Pedersen. We first detail the example of the Heisenberg group of a finite dimensional symplectic vector space. This example, while being the easiest (abelian groups aside), showcases most of the features and issues of graded Lie groups. The only thing it misses is the difference between Puckanzky and Pedersen's stratifications (this difference however needs more complicated groups to be showcased so we won't insist on it).

2.3.1 The Heisenberg group

Let (V,ω) be a symplectic vector space of dimension $2n$. The unitary irreducible representations of $\mathrm{Heis}(V,\omega)$ are of two kinds:

- the one dimensional ones, i.e. the characters. They are induced by the elements of the dual vector space $\xi \in V^*$ through the formula $e^{\mathrm{i}\xi(\cdot)}$.
- the infinite dimensional ones. They are parameterized by $\mathbb{R}^*$, the unique unirep π corresponding to $\lambda \in \mathbb{R}^*$ is characterized by the fact that $\mathrm{d}\pi(Z) = \mathrm{i}\lambda\,\mathrm{Id}$ where Z denotes the basis element of $\mathbb{R}$ in $\mathfrak{heis}(V,\omega) \cong V \oplus \mathbb{R}$.

Let us describe more explicitly the representation corresponding to some $\lambda > 0$ using the so-called Bargmann-Fock representation. Let $J \in \mathrm{End}(V)$ be a compatible complex structure (i.e. ω is J-invariant and $\omega(J\cdot,\cdot) > 0$). We split $V\otimes\mathbb{C}$ according to the $\pm$i-eigenspaces of J: $V\otimes\mathbb{C} = V^{1,0}\oplus V^{0,1}$ and thus also have $\mathfrak{t}_H M\otimes\mathbb{C} = V^{1,0}\oplus V^{0,1}\oplus\mathbb{C}$. The space $V\otimes\mathbb{C}$ is endowed with the hermitian form $\langle\cdot,\cdot\rangle = \mathrm{i}\omega(\cdot,\bar{\cdot})$ for which the previous decomposition is orthogonal and the form is positive (resp. negative) on $V^{1,0}$ (resp. on $V^{0,1}$). Let $\bigoplus_{k=0}^{\infty} Sym^k(V^{1,0}) \subset L^2(V^{0,1},\mathrm{d}\mu)$ be the space of polynomial functions on $V^{0,1}$ with $\mathrm{d}\mu(z) = \pi^{-n}e^{\langle z,z\rangle}\,\mathrm{d}z$ and denote by $\mathcal{H}_+^{BF}$ its closure, i.e. the space of entire functions on $V^{0,1}$. Let $(W_j)_{1\le j\le n}$ be an orthonormal basis of $V^{1,0}$, we have the relations $[W_j,W_k] = [\overline{W_j},\overline{W_k}] = 0$ and $[W_j,\overline{W_k}] = \frac{1}{\mathrm{i}}\delta_{jk}Z$. We can then define the representation π_λ on $\mathcal{H}^{BF}$ as follow :

$$\begin{cases} \mathrm{d}\pi_\lambda(W_j) = \sqrt{\lambda}\mathrm{i}z_j \\ \mathrm{d}\pi_\lambda(\overline{W_j}) = \sqrt{\lambda}\mathrm{i}\dfrac{\partial}{\partial z_j} \\ \mathrm{d}\pi_\lambda(Z) = \mathrm{i}\lambda\,\mathrm{Id} \end{cases}$$

we then have $\forall\ W \in \mathfrak{heis}(V,\omega)\otimes\mathbb{C}, \mathrm{d}\pi_\lambda(W)^* = \mathrm{d}\pi_\lambda(\overline{W})$ and $\mathrm{d}\pi_\lambda$ maps elements of $\mathfrak{heis}(V,\omega)$ to skew-adjoints unbounded operators and hence gives a unitary representation $\pi_\lambda\colon \mathrm{Heis}(V,\omega)\to\mathcal{U}(\mathcal{H}^{BF})$.
A theorem of Stone and Von Neumann ensures that π_λ is an unirep and that every unirep is either a character or isomorphic to π_λ for some $\lambda\in\mathbb{R}^*$.
For $\lambda<0$, π_λ corresponds to the representation $\pi_{|\lambda|}$ of $\mathrm{Heis}(V,-\omega)$ so we have to take $-J$ instead of J and hence swap the roles of $V^{1,0}$ and $V^{0,1}$ so that π_λ acts on $\mathcal{H}^{BF}_- = Hol(V^{1,0})$.

Example 2.3.1. *Let us be more explicit for the standard Heisenberg matrix groups* $\mathrm{Heis}_{2n+1} = \mathrm{Heis}(\mathbb{R}^{2n},\omega_{st})$ *where* $\omega_{st} = \sum_{j=1}^{n}\mathrm{d}x_j\wedge\mathrm{d}y_j$. *We take* $X_1,\cdots,X_n,Y_1,\cdots,Y_n$ *the canonical Darboux basis of* $\mathbb{R}^{2n}$. *The Lie algebra* $\mathfrak{heis}_{2n+1}$ *is generated by* $X_1,\cdots,X_n,Y_1,\cdots,Y_n,Z$ *with relations*

$$\forall i,j, [X_i,Y_j]=\delta_{i,j}Z,$$

and other brackets being equal to 0. *Let* $J\colon\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ *be the linear map defined by:*

$$\begin{cases} J(X_j) = -Y_j \\ J(Y_j) = X_j. \end{cases}$$

It is a complex structure, compatible with ω_{st}. *An orthonormal basis of* $(\mathbb{R}^{2n})^{(1,0)}$ *is then given by* $\left(\frac{1}{\sqrt{2}}(X_j+\mathrm{i}Y_j)\right)_{1\le j\le n}$ *and of* $(\mathbb{R}^{2n})^{(0,1)}$ *by their conjugates* $\left(\frac{1}{\sqrt{2}}(X_j-\mathrm{i}Y_j)\right)_{1\le j\le n}$.

In terms of quantum physics, the operators $\sqrt{\lambda}z_j = \pi_\lambda(\frac{1}{\mathrm{i}}W_j)$ correspond to creation operators and $\sqrt{\lambda}\frac{\partial}{\partial z_j} = \pi_\lambda(\frac{1}{\mathrm{i}}\overline{W_j})$ to annihilation operators. Let us define:

$$H_\lambda = \lambda\sum_{j=1}^{n} z_j\frac{\partial}{\partial z_j} + \frac{1}{2}n\lambda = \mathrm{d}\pi_\lambda(\sum_{j=1}^{n} -\overline{W_j}W_j - \frac{1}{2}\mathrm{i}nZ),$$

the quantum harmonic oscillator (with Plank's constant $\hbar=\lambda$). It is a positive, self-adjoint diagonalizable operator with eigenvectors

$$e_{k_1,\cdots,k_n} = \prod_{\ell=1}^{n}\frac{z_\ell^{k_\ell}}{\sqrt{k_\ell!}}.$$

The action of creations and annihilation operators is then:

$$\begin{cases} \mathrm{d}\pi_\lambda(\frac{1}{\mathrm{i}}W_j)e_{k_1,\cdots,k_n} = \sqrt{\lambda(k_j+1)}e_{k_1,\cdots,k_j+1,\cdots,k_n} \\ \mathrm{d}\pi_\lambda(\frac{1}{\mathrm{i}}\overline{W_j})e_{k_1,\cdots,k_n} = \sqrt{\lambda k_j}e_{k_1,\cdots,k_j,\cdots,k_n} \\ H_\lambda e_{k_1,\cdots,k_n} = \lambda\sum_{\ell=1}^{n}(2k_\ell+1)e_{k_1,\cdots,k_n}. \end{cases}$$

One can show that $\pi_\lambda(C^*(\mathrm{Heis}(V,\omega))) = \mathcal{K}(\mathcal{H}^{BF}_{sgn(\lambda)})$ so that, thanks to a theorem of Lee [58], one can view $C^*(\mathrm{Heis}(V,\omega))$ as a continuous field of C^*-algebras over $\mathbb{R}_+$ with fiber at $\lambda \neq 0, \mathcal{K}(\mathcal{H}^{BF}_+) \oplus \mathcal{K}(\mathcal{H}^{BF}_-)$ and fiber at $0, C^*(V) = C_0(V^*)$. In particular, $C^*(\mathrm{Heis}(V,\omega))$ sits in the exact sequence:

$$0 \longrightarrow C_0(\mathbb{R}_+^*, \mathcal{K}\oplus\mathcal{K}) \longrightarrow C^*(\mathrm{Heis}(V,\omega)) \longrightarrow C_0(V^*) \longrightarrow 0. \tag{2.2}$$

Taking the kernel $C_0^*(\mathrm{Heis}(V,\omega)) \triangleleft C^*(\mathrm{Heis}(V,\omega))$ of the trivial representation gives the exact sequence:

$$0 \longrightarrow C_0(\mathbb{R}_+^*, \mathcal{K}\oplus\mathcal{K}) \longrightarrow C_0^*(\mathrm{Heis}(V,\omega)) \longrightarrow C_0(V^*\setminus 0) \longrightarrow 0. \tag{2.3}$$

We can see $C_0(\mathbb{R}_+^*, \mathcal{K}\oplus\mathcal{K}) = (\mathcal{K}\oplus\mathcal{K}) \rtimes \mathbb{R}$ and $C_0(V^*\setminus 0) \cong C(\mathbb{S}^*V) \rtimes \mathbb{R}$ (the last action is trivial). We therefore also want to have $C_0^*(\mathrm{Heis}(V,\omega))$ in the form $A \rtimes \mathbb{R}$ for some C^*-algebra A endowed with some $\mathbb{R}$-action. This will be the goal of the toy model at the beginning of chapter 3.

Alternatively we can see $C_0^*(\mathrm{Heis}(V,\omega))$ (and $C^*(\mathrm{Heis}(V,\omega))$) as a continuous field of C^* algebras over $\mathbb{R}$ with fiber at $\lambda \neq 0, \mathcal{K}(\mathcal{H}^{BF}_{sgn(\lambda)})$ and the same fiber at 0 than previously. In this setting, we consider $I_\pm$ the restrictions of fields over $\mathbb{R}_\pm$ (they can be seen as quotients of $C_0^*(\mathrm{Heis}(V,\omega))$). They sit in the same exact sequence

$$0 \longrightarrow C_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow I_\pm \longrightarrow C_0(V^*\setminus 0) \longrightarrow 0. \tag{2.4}$$

Obviously we can recover $C_0^*(\mathrm{Heis}(V,\omega))$ from them via the fibered product:

$$C_0^*(\mathrm{Heis}(V,\omega)) \cong I_+ \oplus_{C_0(V^*\setminus 0)} I_-.$$

2.3.2 Kirillov's theory

Kirillov's theory is a powerful tool to study the representation theory of certain classes of Lie groups. For nilpotent Lie groups it gives a homeomorphism between the space of its irreducible unitary representations with the hull-kernel topology and the space of the coadjoint orbits with the quotient topology induced by the euclidean topology on the dual Lie algebra.

In this section G denotes a connected Lie group, $\mathfrak{g}$ its Lie algebra. Recall that the set $\hat{G}$ of irreducible unitary representations of G is canonically identified with the one of its C^*-algebra $C^*(G)$, $\widehat{C^*(G)}$. The unitary dual of a C^*–algebra is endowed with a canonical topology, namely the hull-kernel (or Jacobson) topology. This topology is defined on the set of prime ideals of the algebra (i.e. the kernels of irreducible unitary representations) and then pulled-back to the set of irreducible representations[19]. Let $X \subset Prim(C^*(G))$ be a subset of the set of prime ideals, its closure is defined as:

$$\overline{X} := \{I \in Prim(C^*(G)) \ / \ \bigcap_{J \in X} J \subset I\}.$$

This operation is idempotent and satisfies Kuratowski's axioms, therefore there is a unique topology on the prime spectrum so that this operation corresponds to the closure operator for the topology see [33].

Denote by Ad the adjoint action of G on its Lie algebra $\mathfrak{g}$. It is defined as $\mathrm{Ad}(g)\xi = \partial_t(g \exp(t\xi) g^{-1})_{|t=0}$ for $g \in G, \xi \in \mathfrak{g}$. The coadjoint action is the dual action on $\mathfrak{g}^*$. Given a coadjoint orbit $\Omega \subset \mathfrak{g}^*$, Kirillov constructs an irreducible unitary representation the following way. First choose any $F \in \Omega$ and a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that F vanishes on $[\mathfrak{h}, \mathfrak{h}]$, $\mathfrak{h}$ has to be maximal for this property[20]. These choices give a character of the corresponding subgroup $H \subset G$:

$$\rho_{F,H} \colon \ \exp(\xi) \mapsto e^{2\pi \mathrm{i} \langle F, \xi \rangle}.$$

Then denote by π_Ω the induced representation: $\pi_\Omega = \mathrm{Ind}_H^G(\rho_{F,H})$.

Recall that if $\rho \colon H \to \mathrm{Aut}(V)$ is a unitary representation of H on a Hilbert space V, the induced representation is constructed as follows. Consider the trivial bundle $G \times V \to G$ equipped with the diagonal H-action. Denote by $\mathbb{V} \to X$ the quotient vector bundle over the associated homogeneous space $X = {}^{G}\!/_{H}$. Then G acts naturally on $\Gamma(X, \Omega_X^{1/2} \otimes \mathbb{V})$ by pullback of sections. This action is unitary for the natural hermitian product on the sections and extends to the L^2-completion of the space of sections, becoming a unitary representation $\mathrm{Ind}_H^G(\rho)$.

Theorem 2.3.2 (Kirillov)**.** *Let G be a connected nilpotent Lie group then:*

- *For a coadjoint orbit $\Omega \subset \mathfrak{g}^*$, the representation π_Ω does not depend on the choices of F and $\mathfrak{h}$ up to unitary equivalence.*

[19] Two unitarily equivalent representations have the same kernel.

[20] We also obtain a representation for a non-maximal $\mathfrak{h}$ but it wont be irreducible.

- *For a coadjoint orbit $\Omega \subset \mathfrak{g}^*$, the representation π_Ω is irreducible.*
- *The map ${}^{\mathfrak{g}^*}\!/_{G} \to \hat{G}$ is a bijection. Moreover it is a homeomorphism for the hull-kernel topology on $\hat{G}$ and the quotient of the euclidean topology on ${}^{\mathfrak{g}^*}\!/_{G}$.*

The proof of this theorem can be found in Kirillov's original article [52] up to the continuity of the inverse map which was proved later by Brown [14]. This orbit method contains other results like which orbits correspond to some naturally constructed representations (tensor products, restrictions, inductions...). The general theory of the orbit method can be found in Kirillov's book [53]. Another result that we will use is the following:

Theorem 2.3.3 (Kirillov)**.** *Let G be a simply connected nilpotent Lie group and $\pi \in \hat{G}$ corresponding to a coadjoint orbit of dimension $2n$*[21]*. The representation π can be realized on $L^2(\mathbb{R}^n)$ so that the subspace of smooth vectors is the Schwartz algebra $\mathscr{S}(\mathbb{R}^n)$ and the image of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ by $\mathrm{d}\pi$ is the algebra of differential operators with polynomial coefficients.*

Example 2.3.4. *On the Heisenberg group, let π_λ be the infinite dimensional representation corresponding to $\lambda \in \mathbb{R}^*$. Then in the Schrödinger model for π_λ on $L^2(\mathbb{R})$ we have $\mathrm{d}\pi_\lambda(X) = \frac{\mathrm{d}}{\mathrm{d}x}$, $\mathrm{d}\pi_\lambda(Y) = 2\mathrm{i}\pi\lambda x$ and $\mathrm{d}\pi_\lambda(Z) = 2\mathrm{i}\pi\lambda$ which gives the above result for H_3.*

2.3.3 Puckanzky and Pedersen's stratifications

In this section we describe the stratification of the orbits of unipotent actions as described in [71, 69]. When applied to the coadjoint action this gives a stratification of the unitary dual by Kirillov's homeomorphism. The stratification then allows to parameterize the representation contained in the same stratum. In particular it gives a parameterized (e.g. in family) version of 2.3.3 for the representations of a same stratum of Pedersen's stratification, which will be useful to decompose the C^*-algebra $C_0^*(G)$. In this section we consider G a (real) nilpotent Lie group, $\mathfrak{g}$ its Lie algebra. We denote by n the dimension of G.

Theorem 2.3.5 (Malcev, Jordan, Hölder)**.** *Let $\mathfrak{g}_1 < \cdots < \mathfrak{g}_k < \mathfrak{g}$ be a sequence of subalgebras of $\mathfrak{g}$ of respective dimension $\dim(\mathfrak{g}_j) = m_j$.*

[21] The coadjoint orbits correspond to the symplectic leaves of a natural Poisson structure on $\mathfrak{g}^*$ and thus are even dimensional.

a) There exists a basis $(X_1, \cdots, X_n)$ of $\mathfrak{g}$ such that:

i - For all m, $\mathfrak{h}_m := \mathrm{Vect}(X_1, \cdots, X_m)$ is a Lie subalgebra of $\mathfrak{g}$

ii - $\forall 1 \leq j \leq k, \mathfrak{h}_{m_j} = \mathfrak{g}_j$

b) If for all $1 \leq j \leq k$ the subalgebra $\mathfrak{g}_j$ is an ideal of $\mathfrak{g}$ we can choose the X_j such that condition i above becomes:

iii - $\forall 1 \leq m \leq n, \mathfrak{h}_m \lhd \mathfrak{g}$.

We will call a basis satisfying conditions *ii* and *iii* above a Jordan-Hölder basis of $\mathfrak{g}$ (sometimes also called a strong Malcev basis, a weak Malcev basis satisfying *i* and *ii* instead). If a basis of $\mathfrak{g}$ is chosen, one can consider the group exponential map as a map $\phi\colon \mathbb{R}^n \to G$ by $\phi(s) := \prod_{i=1}^{n} \exp(s_i X_i)$. In this regard Jordan-Hölder basis (and Malcev basis in general) have special properties.

Proposition 2.3.6. *Let $X_1, \cdots, X_n$ be a Jordan-Hölder basis of $\mathfrak{g}$. Denote by $\phi\colon \mathbb{R}^n \to G$ the corresponding exponential map, then:*

- *There exist $P_j \in \mathbb{R}[s_1, \cdots, s_n], 1 \leq j \leq n$ such that*

$$\forall s \in \mathbb{R}^n, \phi(s) = \exp(\sum_{j=1}^{n} P_j(s) X_j).$$

- *$P_j(s) = s_j + Q_j(s)$ where $Q_j \in \mathbb{R}[s_{j+1}, \cdots, s_n]$.*
- *$\log \circ \phi = (P_1, \cdots, P_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial diffeomorphism with polynomial inverse.*
- *If $G_k = \exp(\mathfrak{g}_k)$ then $G_k = \phi(\mathbb{R}^k \times \{0_{n-k}\})$.*

the last two points remain true for general Malcev basis.

Proof. The first point is a direct consequence of the BCH formula for the product of exponentials. The other points are proved by induction by looking at $\mathfrak{g}/\mathfrak{g}_1$ and going back to $\mathfrak{g}$ using the fact that $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}_1$, see [23]. □

This theorem gives us two kinds of coordinates on G. With any basis $(X_1, \cdots, X_n)$ of $\mathfrak{g}$ we have the exponential coordinates:

$$t \mapsto \exp(\sum_{i=1}^{n} t_i X_i).$$

If $(X_1, \cdots, X_n)$ is a Malcev basis there are also Malcev coordinates:

$$s \mapsto \prod_{i=1}^{n} \exp(s_i X_i).$$

The diffeomorphism of $\mathbb{R}^n$ obtained by changing coordinates is polynomial with polynomial inverse. The same holds true for two different Malcev bases. In particular this gives an intrinsic meaning to the Schwartz algebra on a nilpotent Lie group introduced earlier.

We now turn to the parameterization of orbits. We first parameterize a single orbit, then all the generic orbits and finally the remaining ones. The degenerate orbits will be separated with two kinds of partitions: one following Puckanszky and a finer one due to Pedersen. Let V be a finite dimensional real vector space and $G \curvearrowright V$ be a unipotent action[22]. For $v \in V$ we denote by $G_v \subset G$ the stabiliser of v and $\mathfrak{g}_v$ the kernel of the linear map $X \mapsto X \cdot v$ defined on $\mathfrak{g}$. We have $\mathfrak{g}_v = \mathrm{Lie}(G_v)$ and also $G_v = \exp(\mathfrak{g}_v)$ because G is nilpotent (in particular G_v is connected). If $V_0 \subset V$ is a vector subspace stable under the G action we define:

$$\begin{aligned} G_{v,V_0} &:= \{g \in G, gv - v \in V_0\} = G_v \mod V_0 \\ \mathfrak{g}_{v,V_0} &:= \{X \in \mathfrak{g}, Xv \in V_0\} = \mathfrak{g}_v \mod V_0. \end{aligned}$$

Theorem 2.3.7 (Chevalley, Rosenlicht)**.** *Let G be a connected Lie group acting unipotently on the vector space V. Let $v \in V$ then there exists $X_1, \cdots X_k \in \mathfrak{g}$ such that*

$$G \cdot v = \left\{ \left(\prod_{i=1}^{k} \exp(t_i X_i) \right) \cdot v, t_1, \cdots, t_k \in \mathbb{R} \right\}.$$

Moreover the map

$$\begin{aligned} \phi \colon \mathbb{R}^k &\to G \cdot v \\ t &\mapsto \left(\prod_{i=1}^{k} \exp(t_i X_i) \right) \cdot v \end{aligned}$$

is a diffeomorphism. In particular the orbit $G \cdot v$ is a closed submanifold of V. In addition if $(e_1, \cdots, e_m)$ is a basis of V such that each subspace $V_j := \mathrm{Vect}(e_{j+1}, \cdots, e_m)$ is G invariant[23]*.For $t \in \mathbb{R}^k$ define*

$$Q(t) = \sum_{j=1}^{m} Q_j(t) e_j := \phi(t),$$

[22]We will only apply it to the coadjoint action $G \curvearrowright \mathfrak{g}^*$ but this choice of presentation allows to emphasize the difference between the elements of $\mathfrak{g}$ and V.

[23]We can always find such a basis by Engel's theorem.

then each Q_j *is a polynomial map. We can partition* $\{1, \cdots, m\}$ *into two subsets* S[24]*,*T *so that if* $S = \{j_1 < \cdots < j_k\}$ *then each* Q_j *only depends on the* t_α *for which* $j_\alpha \leq j$*. More precisely for* $1 \leq \alpha \leq k$ *we have* $Q_{j_\alpha} = t_{j_\alpha} + P_{j_\alpha}$ *where* $P_{j_\alpha} \in \mathbb{R}[t_1, \cdots, t_{\alpha-1}]$*.*

Proof. Let us fix a basis $(e_1, \cdots, e_m)$ of V as in Engel's theorem. Denote by $\mathfrak{g}_j = \mathfrak{g}_{v,V_j}$. We have $\mathfrak{g}_0 = \mathfrak{g}_1 = \mathfrak{g}$ as the action is unipotent and $\mathfrak{g}_j \subset \mathfrak{g}_{j-1}$. We define the set S as

$$S = \{1 \leq j \leq m, \mathfrak{g}_{j-1} \supsetneq \mathfrak{g}_j\}.$$

We enumerate the elements of S in ascending order $2 \leq j_1 < \cdots < j_k$. For each α we choose $X_\alpha \in \mathfrak{g}_{j_\alpha - 1}$ such that $X_{j_\alpha} v \equiv e_{j_\alpha} \mod V_{j_\alpha}$. Such elements are uniquely determined modulo $\mathfrak{g}_{j_\alpha}$. Induction on $m = \dim(V)$ then shows that $G \cdot v = \left(\prod_{\alpha=1}^{k} \exp(\mathbb{R} X_\alpha) \right) \cdot v$. We now prove the claim about the polynomials Q_j. If $j \leq j_\alpha$ then $\prod_{i=\alpha+1}^{k} \exp(t_i X_i) \cdot v \equiv v \mod V_j$ so $\prod_{i=1}^{k} \exp(t_i X_i) \cdot v \equiv \prod_{i=1}^{\alpha} \exp(t_i X_i) \cdot v \mod V_j$ and thus $Q_j(t)$ only depends on $t_1, \cdots, t_\alpha$. Specifically for $j = j_\alpha$ we get:

$$\begin{aligned} \prod_{i=1}^{\alpha} \exp(t_i X_{j_i}) \cdot v &= \prod_{i=1}^{\alpha-1} \exp(t_i X_{j_i}) \cdot (v + t_\alpha e_{j_\alpha}) \\ &= \left(\prod_{i=1}^{\alpha-1} \exp(t_i X_{j_i}) \cdot v \right) + t_\alpha e_{j_\alpha} \end{aligned}$$

and thus the claim on Q_{j_α} holds. If $V_S = \mathrm{Vect}(e_j, j \in S)$, $p_S \colon V \to V_S$ the projection parallel to $V_T = \mathrm{Vect}(e_j, j \in T)$. We can recursively get $(t_1, \cdots, t_k)$ from $(Q_{j_1}(t), \cdots, Q_{j_k}(t))$ and thus $p_S \circ Q \colon \mathbb{R}^k \to V_S$ is a (polynomial) diffeomorphism (with polynomial inverse). Therefore we obtain that ϕ is a diffeomorphism and that $G \cdot v$ is a submanifold of V (it is actually an algebraic variety). □

There is an issue with the polynomials Q_j constructed above: they depend on v and not on the orbit. To avoid this, one can replace them by polynomials denoted by P_j with the same properties except that:

$$\forall 1 \leq \alpha \leq k, P_{j_\alpha}(u) = u_\alpha.$$

[24] Will be referred after as the set of jump indices.

To do that we let $u_\alpha = Q_{j_\alpha}(t)$ and express recursively the t_α in terms of the u_α. We have proved that $p \circ Q$ had a polynomial inverse and thus the P_j obtained by writing $P(u) = Q(t)$ are still polynomials.

More geometrically, splitting V as $V_S \oplus V_T$ we obtain that for $u \in \mathbb{R}^k$, $P(u)$ is the unique point $w \in G \cdot v$ such that $p_S(w) = (u_1, \cdots, u_k)$. The other $P_j(u), j \in V_T$ are then the remaining coordinates, i.e. $p_T(w)$. The orbit $G \cdot v$ is thus the graph of the polynomial map:

$$\begin{aligned} V_S &\to V_T \\ u &\mapsto (P_j(u))_{j \in T} \end{aligned}$$

We now want to extend this parameterization to multiple orbits at the same time. One could not hope to treat every orbit at the same time (unless the orbit space is Haussdorff) so we need to regroup orbits with similar properties. Their dimension is an important information but also their dimension in every quotient space ${}^{V}\!/\!{}_{V_j}$ which is why we will emphasize the role of the set S used above. For $v \in V$, recall that we have a decreasing sequence of ideals $\mathfrak{g}_j(v) := \mathfrak{g}_{v,V_j}$. The algebra ${}^{\mathfrak{g}_j(v)}\!/\!{}_{\mathfrak{g}_{j+1}(v)}$ maps $\mathbb{R}v$ to a one dimensional subspace. The kernel of this map is trivial and thus the drop of dimension between $\mathfrak{g}_j(v)$ and $\mathfrak{g}_{j+1}(v)$ is either 0 or 1. We set

$$d_j(v) = \dim(G \cdot (v \mod V_j)) = n - \dim(\mathfrak{g}_j(v)), d_j = \max_{v \in V} d_j(v),$$

and $U_j = \{v \in V, d_j(v) = d_j\}$.

Lemma 2.3.8. *The sets U_j are Zariski open.*

Proof. Define $T_j \colon V \to \mathcal{L}(\mathfrak{g}, {}^{V}\!/\!{}_{V_j})$ as $T_j(v)X = X \cdot v \mod V_j$. The map T_j is polynomial and U_j is the set on which T_j is of maximal rank, it is thus Zariski open. □

We now define $S = \{j_1 < \cdots < j_k\}$ to be the set of indices where $d_j \neq d_{j-1}$, T its complement in $\{1, \cdots, m\}$ and $U = \bigcap_{j=1}^{m} U_j = \bigcap_{j \in S} U_j$. The set U is Zariski open and G invariant.

Theorem 2.3.9. *Let $G \curvearrowright V$ be a unipotent action, $(e_1, \cdots, e_m)$ a Jordan-Hölder basis for V, $\{1, \cdots, m\} = S \sqcup T$ the partition constructed above. We identify an element $v \in V$ with its coordinates $(x_j)_{1 \leq j \leq m}$ in the chosen basis. There exists functions $Q_1, \cdots, Q_m \colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ with $k = |S|$ such that:*

- *The functions Q_j are non-singular rational on $U \times \mathbb{R}^k$. For $x \in U$ fixed they are polynomial in $t \in R^k$.*

- *For every* $v = \sum_{j=1}^{m} x_i e_i \in U$ *the map* $t \mapsto Q(t) := \sum_{j=1}^{m} Q_j(x,t)e_j$ *is a diffeomorphism of* $\mathbb{R}^k$ *onto* $G \cdot v$.
- *For* $x \in U$ *fixed,* $Q_j(x,t)$ *only depends on the* t_α *for which* $j_\alpha \leq j$.
- *If* $j \notin S$ *then* $Q_j(x,t) = x_j + R(x_1, \cdots x_{j-1}, t_1, \cdots, t_\alpha)$.
  Here $\alpha = \max\{i, j_i < j\}$ *and* R *is rational. Moreover* $Q_1(x,t) = x_1$.
- *For* $1 \leq \alpha \leq k$, $Q_{j_\alpha}(x,t) = t_\alpha + x_{j_\alpha} + R(x_1, \cdots x_{j_\alpha - 1}, t_1, \cdots, t_{\alpha-1})$.

Proof. We need to mimic the proof of Chevalley-Rosenlicht theorem. To do this we need a coherent (i.e. rational) choice of the $X_\alpha, 1 \leq \alpha \leq k$. Let us fix a basis $(X_1, \cdots, X_n)$ of $\mathfrak{g}$. Denote by $M_j(v)$ the matrix of $T_j(v)$ between this basis and the basis $(e_1, \cdots, e_j)$ of V/V_j. The matrix $M_j(v)$ is the sub-matrix of $M(v) := M_m(v)$ corresponding to the j first rows. The entries of $M(v)$ are polynomials in the coordinates of v and since V_j is G-invariant then the coordinates of $M_j(v)$ only depend on the j first ones of v. We have $\mathrm{rk}(M_j(v)) = d_j(v)$ and thus $\mathrm{rk}(M_{j_\alpha}(v)) = \alpha$ and $\mathrm{rk}(M_j(v)) < \alpha$ for $j < j_\alpha$. We then obtain that $j \in S$ if and only if the equation

$$M_j(v)w = (0, \cdots, 0, 1) \in \mathbb{R}^j,$$

admits a solution. Having such solution is equivalent to the existence of $X_j(v) \in \mathfrak{g}_{j-1}(v) \setminus \mathfrak{g}_j(v)$ such that $X_j(v)v \equiv e_j \mod V_j$. Since the solution is not unique, in order to make a rational choice we have to choose the (maximal) invertible sub-matrix of $M_j(v)$ that will allow to solve the equation. For $1 \leq \alpha \leq k$ we fix a $\alpha \times \alpha$ size sub-matrix L_α of M_{j_α}, independently of v. We make such a choice for each α and gather them in $\theta := (L_1, \cdots, L_k)$. To such θ we associate $U_\theta = \bigcap_{i=1}^{k} \{v \in U, \det(L_i(v)) \neq 0\}$. The U_θ are Zariski open subsets (not G-invariant in general) and cover U. Now on each U_θ one can invert the matrices L_α and obtain $X_j(v)$ and the functions Q_j as in the proof of Chevalley-Rosenlicht's theorem. In order to glue the polynomials obtained from two different choices of θ we need to do what was described before and define the polynomials P_j. $\square$

We now state the geometric consequence of this theorem on the structure of the space of generic orbits U/G.

Corollary 2.3.9.1. *With the same notations as before we have:*

i) Every G*-orbit in* U *meets* V_T *at exactly one point. In particular* $U \cap V_T$ *is a non-empty, Zariski open subset of* V_T.

There is a map $\psi\colon (U \cap V_T) \times V_S \to U$ *such that*

ii) ψ *is a rational, non-singular, bijective map with its inverse also rational and non-singular.*

iii) For every $v \in U \cap V_T$ *the map* $P_v = p_T(\psi(v,\cdot))\colon V_S \to V_T$ *is polynomial, its graph is the orbit* $G \cdot v$.

Proof. Like for a single orbit we take the polynomials P_j that only depends on the orbits (they depend rationally on the point in U). Then $P(v,u)$ is the unique point $w \in G \cdot v$ such that $p_S(w) = u$. We have $w \in U \cap V_T$ if and only if $u = 0$ which proves the first point. Now let

$$\psi\colon (v,u) \mapsto P(v,u) = \sum_{j=1}^{m} P_j(v,u)e_j.$$

It is a rational non singular map and the lines above prove that its inverse is given by $\psi^{-1}(v) = (P(v,0), p_S(v))$ which is also a rational non-singular map. The last claim has already been proved. □

Now to tackle the remaining orbits we need to classify them. Their set S of jump of indices will provide a first way to do so: this is Puckanszky's stratification. Define $d(v) := (d_1(v), \cdots, d_m(v))$ and $\mathcal{D} := \{d(v), v \in V\}$. To $d \in \mathcal{D}$ we associate the partition of $\{1, \cdots, m\}$ given by the set of jump indices $S(d) = \{j, d_j = d_{j-1} - 1\}$ and its complement $T(d)$. The set $\mathcal{D}$ is a subset of $\{1, \cdots, m\}^m$ and we put on it the induced lexicographic order $\succ$, i.e. $d' \succ d \Leftrightarrow \forall j, d'_j \geq d_j$. The partially ordered set $(\mathcal{D}, \succ)$ has a unique maximal element $d^{(1)}$, it is the d corresponding to generic orbits studied above. We define $\mathcal{D}_1 = \{d^{(1)}\}$ and inductively $\mathcal{D}_{i+1}$ to be the set of maximal elements in $\mathcal{D} \setminus \left(\bigcup_{\ell \leq i} \mathcal{D}_\ell \right)$. We then enumerate the elements of $\mathcal{D}$ starting from $\mathcal{D}_1$ and of each $\mathcal{D}_\ell$ successively (we take an arbitrary order for the elements of a same $\mathcal{D}_\ell$). We then get a total order $d^{(1)} > d^{(2)} > \cdots$ on $\mathcal{D}$.

Theorem 2.3.10 (Puckanszky)**.** *Let* $G \curvearrowright V$ *be a unipotent action and fix a Jordan-Hölder basis* $(e_1, \cdots, e_m)$ *of* V. *We define* $\mathcal{D}, S, T$ *as previously, then:*

i) The set $W_d = \bigcup_{d' \geq d} U_{d'}$ *is* G*-invariant and Zariski open in* V. *In particular the set* U_d *is algebraic.*

ii) For $d \in \mathcal{D}$, $S(d) = \{j_1 < \cdots < j_k\}$, then there exists rational functions $P_1, \cdots, P_m : U_d \times \mathbb{R}^k \to \mathbb{R}$[25]*, polynomial in the second variable such that:*

a - $P(v, \cdot) : \mathbb{R}^k \to G \cdot v$ is a diffeomorphism.

b - For a fixed $v \in U_d$, $P_j(v, u)$ only depends on the u_α for which $j_\alpha \leq j$.

c - Identifying v with its coordinates x we have that if $j \notin S(d)$ then

$$P_j(x, t) = x_j + R(x_1, \cdots x_{j-1}, u_1, \cdots, u_\alpha).$$

Here $\alpha = \max\{i, j_i < j\}$ and R is rational.
Moreover $P_1(x, t) = x_1$.

d - For $1 \leq \alpha \leq k$ we have $P_{j_\alpha}(u) = u_\alpha$.

In particular for a fixed $v \in U_d$, $P(v, u)$ only depends on the orbit $G \cdot v$, it is the unique point $w \in G \cdot v$ such that $p_{S(d)}(w) = u$.

iii) Every G-orbit in U_d meets $V_{T(d)}$ in a unique point. The set

$$\Sigma_d := U_d \cap V_{T(d)}$$

is algebraic and is a cross-section for $U_d \twoheadrightarrow U_d/G$. Consequently, the set $\Sigma := \sqcup_{d \in \mathcal{D}} \Sigma_d$ is a cross section of $V \twoheadrightarrow V/G$.

iv) For every $d \in \mathcal{D}$ there exists a birationnal non-singular bijective map $\psi_d : \Sigma_d \times V_{S(d)} \to U_d$ such that for all $v \in \Sigma_d$,

$$p_{T(d)}(\psi_d(v, \cdot)) : V_{S(d)} \to V_{T(d)}$$

is polynomial and its graph is $G \cdot v$.

Proof. We first show i which is the most important point: $W_d = \bigcup_{d' \geq d} U_{d'}$ is Zariski open in V. Let $\ell \in \mathbb{N}$, we consider $W_\ell = \bigcup_{i \leq \ell} U_{d^{(i)}}$. We define

$$S_\ell := \bigcap_{j=1}^{m} \{v \in V, d_j(v) \geq d_j^{(\ell)}\}.$$

[25]If $X \subset V$ is algebraic, a rational function on X is a function $f : X \to \mathbb{R}$ such that there is an open cover $X \subset \cup_\theta U_\theta$ by Zariski open subsets and rational functions (in the usual sense) $f_\theta : U_\theta \to \mathbb{R}$ that coincide with f on each $X \cap U_\theta$.

The set S_ℓ is a finite intersection of Zariski open subsets and is thus Zariski open itself. Moreover we have $U_{d^{(j)}} \subset S_j$ and thus $W_\ell \subset \bigcup_{j\leq \ell} S_j$. We now show the other inclusion which will prove that W_ℓ is Zariski open. Since the sets W_i are nested we just need to prove that $S_i \subset W_i$ for every i. Let us fix i, let k be the integer such that $d^{(i)} \in \mathcal{D}_k = \{d^{(p)}, \cdots, d^{(q)}\}$. Let $v \in S_i$, we have $v \in U_{d^{(r)}}$ for a certain r. For every j we have $d_j^{(r)} = d_j(v) \geq d_j^{(i)}$ and thus $d^{(r)} \succ d^{(i)}$. Now consider the case where $r \geq p$, then by construction of $\mathcal{D}_k$ we have $r = i$ so that $v \in U_{d^{(i)}} \subset W_i$. Otherwise we have $r < p$ and then:

$$v \in U_{d^{(i)}} \cup \bigcup_{d \in \cup_{j\leq k-1}\mathcal{D}_j} U_d = U_{d^{(i)}} \cup \bigcup_{j<p} U_{d^{(j)}} \subset \bigcup_{j\leq i} U_{d^{(j)}} = W_i.$$

The proof of the second point is similar to the previous results. We fix $d \in \mathcal{D}$, $S(d) = \{j_1 < \cdots < j_k\}$, and a basis $(X_1, \cdots, X_n)$ of $\mathfrak{g}$. With the same notations as in the parameterization of generic orbits $\theta = (L_1, \cdots, L_k)$ denotes a sequence of sub-matrices: L_α is a $\alpha \times \alpha$ submatrix of M_{j_α}. We cover U_d by the Zariski open sets $U_\theta = \{v \in V, \det(L_\alpha(v)) \neq 0\}$. As for the generic orbits we can invert the matrices L_α on each U_θ to construct the rational functions Q_j with the required properties on $(U_d \cap U_\theta) \times \mathbb{R}^k$. We cannot however get the polynomials P_j because the desired properties are not valid on the whole $U_\theta \times \mathbb{R}^k$. One needs to modify the Q_j by getting rid of certain coefficients to be able to glue them on $U_d \times \mathbb{R}^k$. Points *iii* and *iv* are similar to the previous corollary. □

Now the goal of this section was to refine Theorem 2.3.3 to multiple orbits at the same time. To do this we now refine Puckanszky's stratification to Pedersen's one. For any stratum of his stratification, Pedersen proved the existence of global canonical coordinates for each orbit in the stratum. By canonical coordinates he means Darboux coordinates for the associated Poisson structure[26]. He gives an explicit algorithm to compute these coordinates and in particular, has a precise control of the denominator (the coordinates are rational maps) which allows to extend 2.3.3.

Let $(X_1, \cdots, X_n)$ be a Jordan-Hölder basis for $\mathfrak{g}$. We denote by

$$G_1 \subset \cdots \subset G_n = G,$$

the sequence of subgroups obtained by exponentiating the ideals

$$\mathfrak{g}_i = \mathrm{Vect}(X_j, j \leq i), i \in \{1, \cdots, n\}.$$

[26]The dual of a Lie algebra carries a natural Poisson manifold for which the coadjoint orbits are the symplectic leaves.

We have also fixed a Jordan-Hölder basis $(e_1, \cdots, e_m)$ for V and denote by $d^i(v)$ the set $d(v)$ for $v \in V$ and the action $G_i \curvearrowright V$. For $v \in V$, denote by $\mathbf{d}(v) = (d^1(v), \cdots, d^n(v))$ and $\mathbf{D} = \{\mathbf{d}(v), v \in V\}$. For $\mathbf{d} \in \mathbf{D}$ we define $U_{\mathbf{d}} = \{v \in V, \mathbf{d}(v) = \mathbf{d}\}$ and obtain a new partition of V:

$$V = \bigsqcup_{\mathbf{d} \in \mathbf{D}} U_{\mathbf{d}}.$$

Moreover since $G_n = G$ we have $d^n(v) = d(v)$ and every layer of Puckanszky's stratification can be decomposed into a finite union of layers of Pedersen's one:

$$\forall d \in \mathcal{D}, U_d = \bigsqcup_{\mathbf{d} \in \mathbf{D}, \mathbf{d}^n = d} U_{\mathbf{d}}.$$

We order $\mathbf{D}$ with the lexicographic order and as for $\mathcal{D}$ we can choose a total order on its elements.

Theorem 2.3.11 (Pedersen)**.** *Let $G \curvearrowright V$ be a unipotent action, we fix Jordan-Hölder bases $(X_1, \cdots, X_n)$ and $(e_1, \cdots, e_m)$ of $\mathfrak{g}$ and V respectively. We define $\mathbf{D}$ as previously. We can order $\mathbf{D}$ so that the sets $W_{\mathbf{d}} := \bigcup_{\mathbf{d}' \geq \mathbf{d}} U_{\mathbf{d}}$ are Zariski open and G-invariant and we can construct rational coordinates for each stratum as before and get a cross section of the map $U_{\mathbf{d}} \twoheadrightarrow {U_{\mathbf{d}}}/{G}$. Moreover if $V = \mathfrak{g}^*$, the G action is the coadjoint action and the basis e is the dual basis of X then the polynomials P_j constructed are Darboux coordinates for the symplectic structure on each orbit of the stratum.*

The proof is similar as before and the details on Darboux coordinates can be found in Pedersen's article [69]. For a stratum $U_{\mathbf{d}}$ we have a corresponding subset $\Lambda_{\mathbf{d}}$ of $\mathfrak{g}^*$ which is a cross section of $U_{\mathbf{d}} \twoheadrightarrow {U_{\mathbf{d}}}/{G}$. Using these coordinates we can prove the following:

Theorem 2.3.12 (Pedersen, Lipsman-Rosenberg)**.** *Let G be a simply connected nilpotent Lie group, $U_{\mathbf{d}}$ a stratum in Pedersen's stratification and $\Lambda_{\mathbf{d}}$ the corresponding cross section. Denote by 2ℓ the common dimension of the orbits in $U_{\mathbf{d}}$. Then one can represent each class of irreducible representations corresponding to orbits in $U_{\mathbf{d}}$ on $L^2(\mathbb{R}^\ell)$ such that the map:*

$$\Lambda_{\mathbf{d}} \times G \to \mathcal{B}(L^2(\mathbb{R}^\ell)$$
$$(\lambda, g) \mapsto \pi_\lambda(g)$$

is continuous where π_λ is the irreducible representation representing the class of the orbit associated to λ. Moreover each one of these representations has the Schwartz algebra as set of smooth vector fields and for $X \in \mathfrak{g}$ the *operator* $\mathrm{d}\pi_\lambda(X)$ *is a differential operator with polynomial coefficients in* X*, its coefficients have a non-degenerate rational dependance on* $\lambda \in \Lambda_{\mathbf{d}}$[27].

Let us denote by $V_1 \subset \cdots \subset V_r = \hat{G}$ Pedersen's stratification of the unitary dual, i.e. $V_i = {}^{W_i}\!/\!_{G}$ where W_i is the i-th stratum in Pedersen's stratification of $\mathfrak{g}^*$. We identify $V_i \setminus V_{i-1}$ with the algebraic set $\Lambda_i \subset \mathfrak{g}^*$. Corresponding to these open subsets are ideals $J_i := \bigcap_{\pi \in \hat{G} \setminus V_i} \ker(\pi)$ of $C^*(G)$. We get an increasing sequence of ideals:

$$\{0\} \lhd J_1 \lhd \cdots \lhd J_r = C^*(G)$$

and by construction $\widehat{{}^{J_i}\!/\!_{J_{i-1}}} = \Lambda_i$. The subquotients thus have Haussdorff spectrum and therefore are continuous fields of C^*-algebras over Λ_i according to [66]. Using the result of Lipsman-Rosenberg stated above we have:

Corollary 2.3.12.1. *For each* $1 \leq i \leq r$ *we have:*

$$ {}^{J_i}\!/\!_{J_{i-1}} \cong \mathscr{C}_0(\Lambda_i, \mathcal{K}_i).$$

Here $\mathcal{K}_i$ *denotes the algebra of compact operators on a separable Hilbert space (of dimension* 1 *for* $i = r$ *and infinite dimensional otherwise).*

Remark 2.3.13. Compare this decomposition with the one given by Puckansky's stratification used in [35]. The later still gives subquotients that are continuous fields of C^*-algebras over the spectrum but we cannot ensure their triviality. Indeed this triviality follows from the results of [61] which apply to Pedersen's stratification. The difference between the two stratifications is not easy to see in low dimension. Examples of graded groups for which the two stratifications differ are given at the end of [12].

We now consider the case where G is a graded Lie group. We thus have the action of inhomogeneous dilations $\delta \colon \mathbb{R}_+^* \to \mathrm{Aut}(G)$.

Proposition 2.3.14. *Let* $(X_1, \cdots, X_n)$ *be a basis of* $\mathfrak{g}$ *consisting of eigenvectors for the dilations, i.e.* $\mathrm{d}\delta_\lambda(X_i) = \lambda^{q_i} X_i$ *where* q_i *is the integer such that* $X_i \in \mathfrak{g}_{q_i}$*. If the* q_i *are in ascending order then* $(X_1, \cdots, X_n)$ *is a Jordan-Hölder basis and the strata of Puckanszky and Pedersen's stratifications are stable by the action of* $\mathbb{R}_+^*$*. Moreover if we remove the trivial representation then the action on each* ${}^{U_d}\!/\!_{G}$ *(d can index strata in both stratification) is free and proper.*

[27] They are even non-degenerate on the complexification of $\Lambda_{\mathbf{d}}$, allowing for holomorphic extension of the matrix coefficients, see [61].

Proof. The fact that X is a Jordan-Hölder basis follows from the fact that $\mathrm{d}\delta_\lambda$ is a Lie algebra homomorphism. The functions $d, S...$ used in the constructions of the strata are clearly invariant under ${}^t\mathrm{d}\delta_\lambda$ and thus each stratum is invariant. The fact that the action is free and proper follows from the identification of ${}^{U_d}\!/_{G}$ with an algebraic subset of $\mathfrak{g}^* \setminus \{0\}$ because we have removed the trivial representation which corresponded to $0_{\mathfrak{g}^*}$. Now the action of ${}^t\mathrm{d}\delta$ on $\mathfrak{g}^* \setminus 0$ is clearly free and proper. □

Corollary 2.3.14.1. *Let G be a graded Lie group, $V_1 \subset \cdots \subset V_r = \hat{G} \setminus \{0\}$ Pedersen's stratification of its unitary dual where the trivial representation is removed. Let $\Lambda_i = V_i \setminus V_{i-1}$ then each set ${}^{\Lambda_i}\!/_{\mathbb{R}_+^*}$ is Hausdorff.*

Proof. It is the quotient of a Hausdorff space by a free and proper action. □

Chapter 3

Crossed products by $\mathbb{R}$-actions

In their article [30], Debord and Skandalis showed an isomorphism between the closure of the C^*-algebra of order 0 pseudodifferential operators and the crossed product of the C^* algebra of the tangent groupoid (minus its trivial representation) by the natural zoom action. The goal here is to generalize this result to filtered manifolds and their filtered calculus. The methods of Debord and Skandalis do not transpose immediately to this new setup. Indeed while the general layout of the proof will be the same, their result relied on the fact that their morphism is an isomorphism at the level of symbols which follows automatically from the commutativity of the symbol algebra. This is no longer the case in the filtered calculus. Our strategy is first to prove that the Debord Skandalis construction gives an isomorphism at the symbol level. To do this we will use Pedersen's stratification of the dual to get as close as possible to things suitable for the commutative case. As a byproduct this decomposition of the symbol algebra will give an analog of Epstein and Melrose's decomposition of the symbol algebra for contact manifolds [34]. This decomposes the symbol algebra into a sequence of nested ideals such that each subquotient is of the form $\mathscr{C}_0(\Lambda, \mathcal{K})$ for a locally compact Haussdorff space Λ and $\mathcal{K}$ the algebra of compact operators on a separable Hilbert space. We first start with a toy model which would correspond to the Heisenberg group but we treat it in a purely functional analysis way using the Toeplitz algebra. We then tackle the isomorphism for symbols on a general bundle of graded Lie groups where we obtain the aforementioned decomposition. Finally we reproduce the reasoning of Debord and Skandalis for the tangent groupoid and get the isomorphism. We also take a different approach in the sense that instead of showing a Morita equivalence $\Psi_H^*(M) \sim_{mor} C_0^*(\mathbb{T}_H^+ M) \rtimes \mathbb{R}_+^*$ (as done in [35]) we will directly construct a map $\Psi_H^*(M) \rtimes \mathbb{R} \to C_0^*(\mathbb{T}_H^+ M)$ and show that it is an isomorphism.

3.1 Toy Model

Before dealing with the general case of contact manifolds of arbitrary dimension, we give a proof of the result for the Heisenberg group H_3 (which is equivalent to the Heisenberg group of any symplectic vector space of dimension 2 after a choice of Darboux basis). The algebra of symbols in this case is linked to the Toeplitz algebra which allows us to showcase the arguments that will be used later on in a very C^*-algebraic way. Recall that if $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ denotes the right shift, the Toeplitz algebra is $\mathcal{T} = C^*(S)$, it is the universal C^*-algebra generated by an isometry and fits into the exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow \mathscr{C}(\mathbb{S}^1) \longrightarrow 0.$$

The creation operators on $C_0^*(H_3)$, once made bounded, will be the wanted $\mathbb{R}_+^*$ family of isometries. We split $C_0^*(H_3)$ into $I_\pm$, allowing us to have such an operator globally defined ($\frac{1}{\mathrm{i}\sqrt{2}}(X \pm \mathrm{i}Y)$ for $I_\pm$). Consider the matrix operator

$$V = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix} \in \mathcal{B}(\ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})),$$

and $A = C^*(V)$, the unital C^*-algebra it generates. If $(e_n)_{n\in\mathbb{N}}$ denotes the canonical Hilbert basis of $\ell^2(\mathbb{N})$ then let $P_{n,m}$ be the rank one operator sending e_n to e_m for $n, m \in \mathbb{N}$, $P_0 = P_{0,0}$ is a projector and we have:

$$1 - VV^* = \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}, 1 - V^*V = \begin{pmatrix} 0 & 0 \\ 0 & P_0 \end{pmatrix}.$$

The relations $P_{n,m} = S^m(1 - SS^*)S^{*n}$ in $\mathcal{B}(\ell_2(\mathbb{N}))$ then imply:

$$V^m(1 - VV^*)V^{*n} = \begin{pmatrix} P_{n,m} & 0 \\ 0 & 0 \end{pmatrix}, V^{*m}(1 - V^*V)V^n = \begin{pmatrix} 0 & 0 \\ 0 & P_{m,n} \end{pmatrix}.$$

The operators $P_{n,m}, n, m \in \mathbb{N}$ generate the C^*-algebra of compact operators $\mathcal{K} = \mathcal{K}(\ell_2(\mathbb{N}))$. We thus have a (closed) ideal $\mathcal{K} \oplus \mathcal{K} \triangleleft A$. Let us now identify the quotient algebra. Let π be the quotient map, $\pi(V)$ generates the quotient algebra. The projectors $1 - VV^*$ and $1 - V^*V$ have finite rank and are hence compact operators. Thus $\pi(V)$ is a unitary operator and the quotient algebra is then canonically isomorphic to the algebra of continuous functions on its spectrum $Sp(\pi(V)) \subset \mathbb{S}^1$. Assume $Sp(\pi(V)) \neq \mathbb{S}^1$ then we have a continuous log on the spectrum and can construct a self-adjoint operator $H \in A$ such that:

$$\pi(V) = \exp(\mathrm{i}\pi(H)) = \pi(\exp(\mathrm{i}H)).$$

This operator decomposes as $H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$ with $H_i^* = H_i$. Therefore $\exp(iH_1) \equiv S \mod \mathcal{K}$ and then $\mathrm{Ind}(S) = \mathrm{Ind}(\exp(\mathrm{i}H_1)) = 0$. However $\mathrm{Ind}(S) = \dim(\ker(S)) - \dim(\ker(S^*)) = -1$ which is a contradiction. We have hence shown that $Sp(\pi(V)) = \mathbb{S}^1$. This gives the exact sequence:

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow A \longrightarrow \mathscr{C}(\mathbb{S}^1) \longrightarrow 0. \tag{3.1}$$

Remark 3.1.1. The same reasoning also shows that A can be identified to the fibered product $\mathcal{T} \oplus_{C(\mathbb{S}^1)} \mathcal{T}$.

We want to identify (2.3) to the sequence obtained from (3.1) after taking a crossed product by $\mathbb{R}$ so first we define a $\mathbb{R}$ action on A, show the sequence obtained from (3.1) is still exact and map it to (2.3).

Let $\alpha\colon \mathbb{R} \to \mathrm{Aut}(A)$ be the family of automorphisms such that α_t sends S to $S_t = \Delta^{\mathrm{i}t} S \Delta^{-\mathrm{i}t}$ where Δ is the unbounded operator sending e_n to $(2n+1)e_n$ (it plays the role of the quantum harmonic oscillator). It is not clear yet that α_t are indeed automorphisms of A. Let us show each α_t acts trivially on the ideal and the quotient, we begin with the following lemma:

Lemma 3.1.2. $\forall t \in \mathbb{R}, S_t \equiv S \mod \mathcal{K}$

The lemma implies that the action is well defined and that it is trivial on the quotient.

Proof. $\alpha_t(P_{n,m})$ is still a rank one operator sending $\mathbb{C}e_n$ to $\mathbb{C}e_m$, more precisely:

$$\begin{aligned} \alpha_t(P_{n,m})(e_n) &= S_t^m P_0 S_t^{*n} e_n \\ &= \Delta^{\mathrm{i}t} S^m P_0 S^{*n} \Delta^{-\mathrm{i}t} e_n \\ &= \left(\frac{m + \frac{1}{2}}{n + \frac{1}{2}} \right)^{\mathrm{i}t} e_m. \end{aligned}$$

From that we see that the action stabilizes the compact operators and hence factors through the quotient. We now prove the lemma. Define $\Psi_t = S_t - S$ and it's finite rank truncations:

$$\Psi_t^{(n)} \colon e_k \mapsto \begin{cases} \Psi_t(e_k) \text{ if } k \leq n \\ 0 \text{ otherwise.} \end{cases}$$

The $\Psi_t^{(n)}$ are of course compact operators so we now show that:

$$\lim_{n \to +\infty} \Psi_t^{(n)} = \Psi_t$$

in the operator norm topology. Let $\Phi_t^{(n)} = \Psi_t - \Psi_t^{(n)}$ we have:

$$\|\Phi_t^{(n)}\|^2 = \|\Phi_t^{(n)*}\Phi_t^{(n)}\| = \rho(\Phi_t^{(n)*}\Phi_t^{(n)}),$$

where ρ is the spectral radius. We can compute the eigenvalues explicitly:

$$\Phi_t^{(n)*}\Phi_t^{(n)}(e_k) = \begin{cases} \left|\left(\dfrac{k+\frac{3}{2}}{k+\frac{1}{2}}\right)^{\mathrm{i}t} - 1\right|^2 e_k \text{ if } k > n \\ 0 \text{ otherwise.} \end{cases}$$

Therefore we have

$$\|\Psi_t^{(n)} - \Psi_t\| = \sup_{k \geq n} \left|\left(\frac{k+\frac{3}{2}}{k+\frac{1}{2}}\right)^{\mathrm{i}t} - 1\right|.$$

We then get the approximation:

$$\begin{aligned} \left|\left(\frac{k+\frac{3}{2}}{k+\frac{1}{2}}\right)^{\mathrm{i}t} - 1\right| &= \left|\exp\left(\mathrm{i}t\left[\ln\left(1+\frac{3}{2n}\right) - \ln\left(1+\frac{1}{2n}\right)\right]\right) - 1\right| \\ &= \left|\exp\left(\frac{\mathrm{i}t}{n} + o\left(\frac{1}{n}\right)\right) - 1\right| \\ &= \mathcal{O}\left(\frac{1}{n}\right). \end{aligned}$$

Here the o and $\mathcal{O}$ can be taken uniformly in t in any compact subset of $\mathbb{R}$. Hence we get the convergence:

$$\lim_{n \to +\infty} \Psi_t^{(n)} = \Psi_t,$$

in the norm operator topology (uniformly for t in any compact subset of $\mathbb{R}$). Thus the operators Ψ_t are compact and we have proven the lemma. □

Corollary 3.1.2.1. *The $\mathbb{R}$ action on A is well defined and trivial on the quotient $\mathscr{C}(\mathbb{S}^1)$.*

Lemma 3.1.3. *The crossed product $\mathcal{K} \rtimes_\alpha \mathbb{R}$ is trivial i.e. isomorphic to the tensor product $\mathcal{K} \otimes \mathscr{C}_0(\mathbb{R}_+^*)$.*

Proof. $\alpha_t = Ad(\Delta^{\mathrm{i}t})$ and $\Delta^{\mathrm{i}t}$ are multipliers of $\mathcal{K}$ so we can define the map :

$$\begin{array}{ccc} \mathcal{K} \otimes_{alg} \mathscr{C}_c(\mathbb{R}) & \to & \mathscr{C}_c(\mathbb{R}, \mathcal{K}) \\ f \otimes T & \mapsto & (s \mapsto f(s)T\Delta^{-\mathrm{i}s}) \end{array}$$

which is an algebra morphism (the later is considered with the product twisted by α) extends to an isomorphism between $\mathcal{K} \otimes \mathscr{C}_0(\mathbb{R}_+^*)$ and $\mathcal{K} \rtimes_\alpha \mathbb{R}$ □

Remark 3.1.4. We used the fact that $\mathbb{R}$ is amenable hence we did not make any difference between the maximal and reduced crossed products by $\mathbb{R}$ and tensor products by $C^*(\mathbb{R})$. We also used the Pontryagin duality:

$$C^*(\mathbb{R}) \cong \mathscr{C}_0(\mathbb{R}_+^*)^{\cdot}$$

Corollary 3.1.4.1. *We have the exact sequence*

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K} \oplus \mathcal{K}) \longrightarrow A \rtimes_\alpha \mathbb{R} \longrightarrow \mathscr{C}_0(\mathbb{R}^2 \setminus 0) \longrightarrow 0. \tag{3.2}$$

Proof. We use the exactness of the maximal tensor product (we can also use the reduced tensor product which is exact since $\mathbb{R}$ is amenable). We have already identified the ideal in the previous lemma. The action on the quotient being trivial we have

$$\mathscr{C}(\mathbb{S}^1) \rtimes_\alpha \mathbb{R} \cong \mathscr{C}(\mathbb{S}^1) \otimes \mathscr{C}_0(\mathbb{R}_+^*) \cong \mathscr{C}_0(\mathbb{S}^1 \times \mathbb{R}_+^*) \cong \mathscr{C}_0(\mathbb{R}^2 \setminus 0).$$

Therefore the exact sequence (3.2) derives from (3.1). □

Remark 3.1.5. Looking at α acting on $\mathcal{T}$ we also get the exact sequence:

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow \mathcal{T} \rtimes_\alpha \mathbb{R} \longrightarrow \mathscr{C}_0(\mathbb{R}^2 \setminus 0) \longrightarrow 0. \tag{3.3}$$

We still have $A \rtimes \mathbb{R} \cong (\mathcal{T} \rtimes_\alpha \mathbb{R}) \oplus_{\mathscr{C}_0(\mathbb{R}^2 \setminus 0)} (\mathcal{T} \rtimes_\alpha \mathbb{R})$.

We now want to relate the exact sequences (3.2) and (2.3). This will show an isomorphism between $A \rtimes_\alpha \mathbb{R}$ and $C_0^*(H_3)$. To do that we show isomorphisms between the exact sequences (3.3) and (2.4) and then take a fibered product. For that we need two morphisms: $\varphi\colon \mathcal{T} \to \mathcal{M}(I_\pm)$[1] and $\beta\colon \mathbb{R} \to \mathcal{M}_u(I_\pm)$ ($\mathcal{M}(\cdot)$ denotes the multiplier algebra and $\mathcal{M}_u(\cdot)$ the unitary elements of this algebra). Those morphisms have to satisfy the relation:

$$\forall t \in \mathbb{R}, \varphi \circ \alpha_t = \mathrm{Ad}(\beta(t)) \circ \varphi. \tag{3.4}$$

Let $W_\pm = \dfrac{1}{\mathrm{i}\sqrt{2}}(X \pm \mathrm{i}Y)$, as an unbounded multiplier of $I_\pm$ it is invertible and we can therefore construct the (bounded) multiplier $\widetilde{W}_\pm = |W_\pm|^{-1} W_\pm$ where $|W| = \sqrt{W^* W}$. In any representation π_λ with $\lambda > 0$ $\mathrm{d}\pi_\lambda(\widetilde{W}_+) e_n = e_{n+1}$ (the same applies for $\widetilde{W}_-$ and $\lambda < 0$) and $\pi_{\mu,\nu}(\widetilde{W}_\pm) = \dfrac{\nu \mp \mathrm{i}\mu}{\sqrt{\mu^2 + \nu^2}}$ for $(\mu, \nu) \in \mathbb{R}^2 \setminus 0$. By construction $\widetilde{W}_\pm$ is an isometry so by universal property of the Toeplitz algebra we have a morphism $\varphi\colon \mathcal{T} \to \mathcal{M}(I_\pm)$ sending S to $\widetilde{W}_\pm$.

[1] Remember $I_\pm$ are ideals of $C_0^*(H_3)$ defined in the previous chapter.

To define β let $H_{\pm} = W_{\pm}^{*}W_{\pm} - \dfrac{\mathrm{i}}{2}Z$. In any representation (of the appropriate sign) it is mapped to the quantum harmonic oscillator. It is thus a positive self-adjoint unbounded multiplier and we can take $\beta(t) = H_{\pm}^{\mathrm{i}t}$ through holomorphic functional calculus. This defines a group homomorphism $\beta\colon \mathbb{R} \to \mathcal{M}_u(I_{\pm})$. The relation (3.4) is then satisfied by construction of Δ and $H_{\pm}$. Indeed, once extended to multipliers, $\varphi(\Delta) = H_{\pm}$. We have thus constructed a homomorphism $\phi \rtimes \beta\colon \mathcal{T} \rtimes_{\alpha} \mathbb{R} \to I_{\pm}$. As it acts fiberwise (if we see $\mathcal{T} \rtimes_{\alpha} \mathbb{R}, I_{\pm}$ as continuous fields of C^{*}-algebras over $\mathbb{R}^{*}$), we get a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathbb{R}_{+}^{*}, \mathcal{K}) & \longrightarrow & \mathcal{T} \rtimes_{\alpha} \mathbb{R} & \longrightarrow & C_0(\mathbb{R}^2 \setminus 0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \varphi\rtimes\beta} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & C_0(\mathbb{R}_{+}^{*}, \mathcal{K}) & \longrightarrow & I_{\pm} & \longrightarrow & C_0(\mathbb{R}^2 \setminus 0) & \longrightarrow & 0.
\end{array}$$

The vertical arrows on the left and on the right are, at the level of $C_c(\mathbb{R}, \mathcal{K})$ and $C_c(\mathbb{R}, \mathbb{S}^1)$ defined by a Fourier transform in the $\mathbb{R}$ variable. They thus extend to Id by Pontryagin duality. The diagram then implies that $\varphi \rtimes \beta$ is an isomorphism by a two-out-of-three lemma. We moreover have a commutative diagram:

$$\begin{array}{ccccc}
 & & \mathcal{T} \rtimes_{\alpha} \mathbb{R} \xrightarrow{\varphi\rtimes_{\alpha}\beta} I_{+} & & \\
 & \nearrow^{pr_1} & & \searrow & \\
A \rtimes_{\alpha} \mathbb{R} & & & & \mathscr{C}_0(\mathbb{R}^2 \setminus 0). \\
 & \searrow_{pr_2} & & \nearrow & \\
 & & \mathcal{T} \rtimes_{\alpha} \mathbb{R} \xrightarrow{\varphi\rtimes_{\alpha}\beta} I_{-} & &
\end{array}$$

Both compositions of arrows on top or down correspond to the natural map $A \rtimes_{\alpha} \mathbb{R} \to C_0(\mathbb{R}^2 \setminus 0)$ of the exact sequence (3.2). We can thus use the universal property of the fibered product to get a map:

$$\Phi\colon A \rtimes_{\alpha} \mathbb{R} \to I_{+} \oplus_{C_0(\mathbb{R}^2\setminus 0)} I_{-} \cong C_0^{*}(H_3).$$

This map is an isomorphism and makes the following diagram commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{C}_0(\mathbb{R}_{+}^{*}, \mathcal{K} \oplus \mathcal{K}) & \longrightarrow & A \rtimes_{\alpha} \mathbb{R} & \longrightarrow & \mathscr{C}_0(\mathbb{R}^2 \setminus 0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \mathscr{C}_0(\mathbb{R}_{+}^{*}, \mathcal{K} \oplus \mathcal{K}) & \longrightarrow & C_0^{*}(H_3) & \longrightarrow & \mathscr{C}_0(\mathbb{R}^2 \setminus 0) & \longrightarrow & 0.
\end{array}$$

3.2 The case of symbols

Let $G \to M$ denote a bundle of graded Lie groups over a compact base. We want to construct an isomorphism

$$\Phi \colon \Sigma(G) \rtimes \mathbb{R} \to C_0^*(G).$$

We first need to construct the $\mathbb{R}$ action on $\Sigma(G)$ underlying the crossed product. For that let $\Delta_0 \in \mathcal{U}(\mathfrak{g})$ be an essentially self-adjoint, positive Rockland symbol of positive even order m. Thanks to Theorem 2.2.31, we can construct the complex powers of Δ_0. We obtain a family of symbols:

$$\Delta_0^{\mathrm{i}t/m} \in \Sigma^{\mathrm{i}t}(G), t \in \mathbb{R},$$

which is a group under composition law for symbols. Since Δ_0 is self-adjoint they also are unitary. Let $\sigma \in \Sigma_c^0(G)$ and $t \in \mathbb{R}$, the symbol

$$\mathrm{Ad}(\Delta_0^{\mathrm{i}t/m})\sigma := \Delta_0^{\mathrm{i}t/m}\sigma\Delta_0^{-\mathrm{i}t/m}$$

is of order 0. We hence obtain an action of $\mathbb{R}$ on $\Sigma_c^0(G)$. This action is continuous and thus defines an action on the C^*-completion $\Sigma(G)$. We now want to define the morphism Φ sending the crossed product to $C_0^*(G)$. To do this we need a (strongly continuous) family of unitary multipliers

$$u_t \in M_u(C_0^*(G)),$$

a morphism

$$\varphi \colon \Sigma(G) \to M(C_0^*(G)),$$

with the property that:

$$\forall t \in \mathbb{R}, \sigma \in \Sigma(G), \varphi\left(\mathrm{Ad}(\Delta_0^{\mathrm{i}t/m})\sigma\right) = u_t\varphi(\sigma)u_t^*. \tag{3.5}$$

These two constructions are similar and rely on a lemma due to Taylor. This lemma's purpose is to represent each class of principal symbols by a particular distribution on G which will be homogeneous on the nose.

Definition 3.2.1. Let $s \in \mathbb{C}$ be a complex number, denote by

$$\mathcal{K}^s(G) \subset \mathscr{S}'(G, \Omega^{1/2})$$

the set of fibered tempered distributions on G that have singular support contained in the unit section and are homogeneous of degree s with respect to the family of dilations $(\delta_\lambda)_{\lambda>0}$.

On the space of Schwartz sections[2] we define a Fourier transform, identifying G with $\mathfrak{g}$, it becomes:

$$\begin{aligned}\mathcal{F}\colon \mathscr{S}(G,\Omega^{1/2}) &\to \mathscr{S}(\mathfrak{g}^*)\\ f &\mapsto \left((x,\eta)\mapsto \int_{\xi\in\mathfrak{g}_x} e^{\mathrm{i}\langle\eta,\xi\rangle} f(x,\xi)\right)\end{aligned}$$

Consider the dual action of $\mathbb{R}_+^*$ on $\mathfrak{g}^*$ given by $({}^t\mathrm{d}\delta_\lambda)_{\lambda>0}$. The Fourier transform has the following equivariance property:

$$\forall\lambda>0, \mathcal{F}\circ\delta_\lambda^* = {}^t\mathrm{d}\delta_\lambda^*\circ\mathcal{F}$$

Among the Schwartz sections, denote by $\mathscr{S}_0(G,\Omega^{1/2})$ those for which the Fourier transform vanishes at infinite order on the zero section of $\mathfrak{g}^*$. It is a closed subalgebra of the Schwartz algebra (for the topology induced by the Schwartz semi-norms).

Lemma 3.2.2 (Taylor [76])**.** *Let $k\in\mathbb{C}$, $u\in S_p^k(G)$. Then there exists a unique smooth function $v\in\mathscr{C}^\infty(\mathfrak{g}^*\setminus 0)$ such that:*

- *v is homogeneous of degree k: $\forall\lambda>0, v\circ{}^t\mathrm{d}\delta_\lambda = \lambda^k v$*
- *if $\chi\in\mathscr{C}_c^\infty(\mathfrak{g}^*)$ is equal to 1 on a neighborhood of the zero section then $\hat{u}-(1-\chi)v\in\mathscr{S}(\mathfrak{g}^*)$.*

Conversely if a function $v\in\mathscr{C}^\infty(\mathfrak{g}^*\setminus 0)$ is homogeneous of degree k then one can find a symbol $u\in S_c^k(G)$ such that the second property is satisfied.

Corollary 3.2.2.1. *There is a linear isomorphim $\Theta\colon\Sigma_p^s(G)\to\mathcal{K}^s(G)$ for each $s\in\{z\in\mathbb{C}\ /\ \Re(z)>-n\}$ which is compatible with the convolution and adjoints. Here n denotes the homogeneous dimension of G.*

Proof. Let $u\in S_p^s(G)$. Define v as in Taylor's lemma. The condition on the order $\Re(s)>-n$ ensures that $v\in L^1_{loc}(\mathfrak{g}^*)$. The function v thus extends to a tempered distribution

$$v\in\mathscr{S}'(\mathfrak{g}^*).$$

Its inverse Fourier transform w is in $\mathscr{S}'(G,\Omega^{1/2})$ and is homogeneous of degree s, i.e. $w\in\mathcal{K}^s(G)$. This construction gives a linear map $S_c^s(G)\to\mathcal{K}^s(G)$ which is surjective. By the uniqueness of v in Taylor's lemma, the kernel of this map consists of smooth symbols. It is thus equal to $S_p^{-\infty}(G)$. The resulting quotient map $\Theta\colon\Sigma_p^s(G)\to\mathcal{K}^s(G)$ is then an isomorphism. □

[2] Defined in the previous chapter.

Lemma 3.2.3 (Christ, Geller, Głowacki, Polin [19])**.** *Let $u \in \mathcal{K}^s(G)$. The convolution by u is a continuous map from $\mathscr{S}_0(G)$ to itself.*

Proposition 3.2.4. *Let $k \in \mathcal{K}^s(G)$ be a homogeneous distribution of degree $s \in \mathbb{C}$. If $\Re(s) \leq 0$ then k extends to an element of $M(C_0^*(G))$. Moreover*

$$\|k\| = \sup_{x \in M} \|k_x\|_{L^2(G_x)} = \sup_{x \in M, \pi \in \hat{G}_x} \|\pi(k_x)\|.$$

Proposition 3.2.5. *The composition of Θ with the injection*

$$\mathcal{K}^0(G) \hookrightarrow M(C_0^*(G))$$

*extends to a *-homomorphism*

$$\varphi \colon \Sigma(G) \to M(C_0^*(G)).$$

Proof. It remains to show that this composition is continuous but this is a consequence of the previous proposition since for $u \in S_c^0(G), x \in M, \pi \in \widehat{G_x}$ we have:

$$\|\pi(\Theta(u)_x)\| \leq \|u\|_{L^2(G)}.$$

Indeed, $\beta := u - \Theta(u) \in \mathscr{S}(G, \Omega^{1/2})$ and by the Plancherel formula:

$$\|\pi \circ \delta_\lambda(\beta)\| \leq \operatorname{Tr}(\pi \circ \delta_\lambda(\beta^* * \beta))\| = \int_{{}^t\delta_\lambda(\mathcal{O}_\pi)} \widehat{\beta^* * \beta}(x, \xi) \, \mathrm{d}\xi \xrightarrow[\lambda \to +\infty]{} 0.$$

The Plancherel formula can be found in [23], here $\mathcal{O}_\pi$ denotes the coadjoint orbit in $\mathfrak{g}^*$ corresponding to the representation π through Kirillov's orbit method. Since $\Theta(u)$ is homogeneous of degree 0 we have:

$$\|\pi(\Theta(u)_x)\| = \|\pi \circ \delta_\lambda(\Theta(u)_x)\|,$$

for every $\lambda > 0$. We thus get:

$$\|\pi(\Theta(u)_x)\| \leq \lim_{\lambda \to +\infty} \|\pi \circ \delta_\lambda(u_x)\| \leq \|u\|.$$

□

In the same way, let us denote by u_t the extension as an element of $M(C_0^*(G))$ of $\Theta(\Delta_0^{it/m})$. Since Θ preserves the adjoints then u_t is an unitary multiplier and since it preserves the convolution then $u_t u_s = u_{t+s}$ and we get the relation (3.5).

We have thus constructed from φ and $(u_t)_{t \in \mathbb{R}}$ a *-homomorphism:

$$\Phi \colon \Sigma(G) \rtimes \mathbb{R} \to C_0^*(G).$$

The rest of this section is devoted to the proof that Φ is an isomorphism.

To show that Φ is an isomorphism, we need a better understanding of the representations of $\Sigma(T_H M)$. We first reduce our study to the bundle of osculating Lie group $T_H M$ to a single graded Lie group. This follows from the following result:

Proposition 3.2.6. *The algebras $\Sigma(G)$ and $C_0^*(G)$ are the sections of continuous fields of C^*-algebras over M. Their respective fibers at $x \in M$ are equal to $\Sigma(G_x)$ and $C_0^*(G_x)$. Moreover the action of $\mathbb{R}$ on $\Sigma(G)$ preserves the fibers and Φ restricts to a family of morphisms from $\Sigma(G_x) \rtimes \mathbb{R}$ to $C_0^*(G_x)$.*

We can now consider a graded Lie group G and prove that

$$\Phi \colon \Sigma(G) \rtimes \mathbb{R} \to C_0^*(G)$$

is an isomorphism.

Since φ maps $\Sigma(G)$ to $M(C_0^*(G))$, every non-trivial irreducible representation of G induces a representation of $\Sigma(G)$. Moreover, since the image of φ consists of invariant elements under the inhomogeneous $\mathbb{R}_+^*$-action then $\pi \circ \varphi = \delta_\lambda(\pi) \circ \varphi$ for all $\lambda > 0$ and $\pi \in \hat{G} \setminus \{1\}$. The following theorem due to Fermanian-Kammerer and Fischer asserts that these representations of $\Sigma(G)$ are irreducible and allow to recover the whole spectrum of $\Sigma(G)$.

Theorem 3.2.7 (Fermanian-Kammerer, Fischer [36])**.** *Let G be a graded Lie group. For $\pi \in \hat{G} \setminus \{1\}$ denote by $[\pi]$ its class in ${}^{(\hat{G} \setminus \{1\})}\!\big/_{\mathbb{R}_+^*}$. For every $\pi \in \hat{G} \setminus \{1\}$, we have $\pi \circ \varphi \in \widehat{\Sigma(G)}$ and the map*

$$\begin{aligned} R \colon {}^{(\hat{G} \setminus \{1\})}\!\big/_{\mathbb{R}_+^*} &\to \widehat{\Sigma(G)} \\ [\pi] &\mapsto \pi \circ \varphi \end{aligned}$$

is a homeomorphism.

Proof. Since our definition of the symbol algebra is different than the one of [36] we detail the proof in our context. Let us prove that R is injective. Let $\pi_1, \pi_2 \in \hat{G} \setminus \{1\}$ be non trivial unirreps of G with $[\pi_1] \neq [\pi_2]$. Since ${}^{(\hat{G} \setminus \{1\})}\!\big/_{\mathbb{R}_+^*}$ is a T_0 space (it is the spectrum of the algebra $C_0^*(G) \rtimes \mathbb{R}_+^*$) then either $[\pi_1] \notin \overline{\{[\pi_2]\}}$ or $[\pi_2] \notin \overline{\{[\pi_1]\}}$[3]. Without loss of generality we will assume the former. We can use a similar reasoning than the one in [42] to construct a function $f \in \mathscr{S}_0(G)$ such that for every $\pi \in \overline{\mathbb{R}_+^* \cdot \pi_2}$

[3]This can also be seen by using Pedersen's stratification.

we get $\pi(f) = 0$ and for every other unirrep π we have $\pi(f)$ positive and left invertible. We can recover a symbol from f with $\sigma = \int_0^{+\infty} \delta_{\lambda *} f \frac{\mathrm{d}\lambda}{\lambda}$. If $\pi \in \hat{G} \setminus \{1\}$ we have:

$$\pi \circ \varphi(\sigma) = \int_0^{+\infty} (\lambda \cdot \pi)(f) \frac{\mathrm{d}\lambda}{\lambda}.$$

Therefore we have $R([\pi_2])(\sigma) = 0$ and $R(\pi_1)(\sigma) \neq 0$ and $R(\pi_1) \neq R(\pi_2)$ (this also shows that the representations obtained through R are all nonzero).

We now prove the surjectivity of R. We proceed as in [36] and construct a right inverse. Let $|\cdot|$ be a homogeneous quasi-norm on G. Take $f \in \mathscr{C}_c^\infty(G, \Omega^{1/2})$ and take its (euclidean) Fourier transform $\hat{f}$. Now the function ${}^t\mathrm{d}\delta^*_{|\cdot|^{-1}} \hat{f} \in \mathscr{C}^\infty(\mathfrak{g}^* \setminus 0)$ is homogeneous of degree 0. By virtue of lemma 3.2.2 it corresponds to a unique symbol $\sigma_f \in \Sigma^0(G)$. Let ρ be a representation of $\Sigma(G)$, $\pi_\rho \colon f \to \rho(\sigma_f)$ extends to a continuous representation of $C^*(G)$. The formula for generalized character of unitary representations of nilpotent Lie groups (see [53, 23]) shows that π_ρ is the same representation as the one constructed in [36] Lemma 5.7. Therefore we have that π_ρ is a non-trivial unirrep of G and the map $\rho \mapsto [\pi_\rho]$ is a right inverse to R. Thus far, we have proven that R was a continuous bijection. Its inverse being the map $\rho \mapsto [\pi_\rho]$ which is also continuous, R is therefore a homeomorphism. □

Corollary 3.2.7.1. *Let (M, H) be a filtered manifold then the spectrum of $\Sigma(T_H M)$ is homeomorphic to* $\left(\widehat{T_H M} \setminus \{1\}\right)/\mathbb{R}_+^*$.

Proof. Thanks to 3.2.6 the result reduces to each fiber over the points of M, we can then apply 3.2.7. □

We are now ready to prove the main theorem:

Theorem 3.2.8. *Let G be a graded Lie group, then*

$$\Phi \colon \Sigma(G) \rtimes \mathbb{R} \to C_0^*(G),$$

is an isomorphism of C^*-algebras.

Corollary 3.2.8.1. *Let (M, H) be a filtered manifold, then*

$$\Phi \colon \Sigma(T_H M) \rtimes \mathbb{R} \to C_0^*(T_H M),$$

is an isomorphism of C^*-algebras.

Proof. Let $= V_0 \subset V_1 \subset \cdots \subset V_r = \hat{G} \setminus \{1\}$ be a fine stratification of $\hat{G}$ as in Theorem 2.3.14. Recall that every V_i is open and $\mathbb{R}_+^*$-invariant, that the spaces $\Lambda_i := V_i \setminus V_{i-1}$ are Hausdorff and that the action $\mathbb{R}_+^* \curvearrowright \Lambda_i$ is free and proper. This stratification induces a filtration of $\Sigma(G)$ and $C_0^*(G)$ into sequences of increasing ideals:

$$\{0\} = J_0 \lhd J_1 \lhd \cdots \lhd J_r = C_0^*(G)$$
$$\{0\} = \Sigma_0 \lhd \Sigma_1 \lhd \cdots \lhd \Sigma_r = \Sigma(G)$$

with $J_i = \bigcap_{\pi \in \hat{G} \setminus V_i} \ker(\pi)$ and $\Sigma_i = \bigcap_{\pi \in \hat{G} \setminus V_i} \ker(\pi \circ \varphi)$. By construction the subsets Σ_i are $\mathbb{R}$-invariant and Φ restricts to maps $\Phi \colon \Sigma_i \rtimes \mathbb{R} \to J_i$. The spectrum of ${J_i}/{J_{i-1}}$ is $\widehat{{J_i}/{J_{i-1}}} = \widehat{J_i} \setminus \widehat{J_{i-1}} = V_i \setminus V_{i-1} = \Lambda_i$ for every i. Recall that the quotient algebra ${J_i}/{J_{i-1}}$ is a continuous field of C^*-algebras over its spectrum Λ_i and by Corollary 2.3.12.1 (see also [9]) the fiber at $\lambda \in \Lambda_i$ is $\mathcal{K}(\mathcal{H}_i)$ where $\mathcal{H}_i$ is a Hilbert space (of dimension 1 for $i = r$ and of infinite dimension otherwise). Since the action of $\mathbb{R}_+^*$ on Λ_i is free and proper, ${\Lambda_i}/{\mathbb{R}_+^*}$ is also a Hausdorff space and each ${\Sigma_i}/{\Sigma_{i-1}}$ is a continuous field of C^*-algebras over ${\Lambda_i}/{\mathbb{R}_+^*}$. The fiber at $[\lambda] \in {\Lambda_i}/{\mathbb{R}_+^*}$ is the image of $\lambda \circ \varphi$. It is a sub-algebra of $\mathcal{B}(\mathcal{H}_i)$, we will denote it by A_λ.

Lemma 3.2.9. *For every $\lambda \in \hat{G} \setminus \{1\}$, A_λ is a simple algebra.*

Proof. The spectrum of a continuous field of C^*-algebra is in bijection with the direct sum of the spectra of the fibers. Here the spectrum of the continuous field is exactly its base hence the fibers have trivial spectrum and are thus simple. Indeed having a trivial spectrum means having no non-trivial primitive ideal but since every ideal is an intersection of primitive ideals then an algebra with no non-trivial primitive ideals is simple. □

Lemma 3.2.10. *For every $\lambda \in \hat{G} \setminus \{1\}$, A_λ contains $\mathcal{K}(\mathcal{H}_i)$ (where $\lambda \in \Lambda_i$).*

Proof. As in the proof of 3.2.7, let us choose a homogeneous quasi-norm on G. This induces a $*$-homomorphim $\mathcal{S}(G) \to \Sigma(G)$ and if $\lambda \in \hat{G}$ is of norm 1 then the previous morphism composed with λ seen as a representation of $\Sigma(G)$ extends to λ seen as a representation of $C_0^*(G)$. Thus A_λ contains $\lambda(C^*(G)) = \mathcal{K}(\mathcal{H}_\lambda)$. □

Corollary 3.2.10.1. *The algebras ${\Sigma_i}/{\Sigma_{i-1}}$ are trivial fields of C^*-algebras:*

$${\Sigma_i}/{\Sigma_{i-1}} \cong \mathscr{C}_0\left({\Lambda_i}/{\mathbb{R}_+^*}, \mathcal{K}(\mathcal{H}_i)\right).$$

Corollary 3.2.10.2. *For every $i \geq 0$, the action of $\mathbb{R}$ on ${}^{\Sigma_i}\!/_{\Sigma_{i-1}}$ is inner and $\Phi\colon {}^{\Sigma_i}\!/_{\Sigma_{i-1}} \rtimes \mathbb{R} \to {}^{J_i}\!/_{J_{i-1}}$ is an isomorphim.*

Proof. Let us fix a number $t \in \mathbb{R}$. Since $\Delta_0^{\mathrm{i}t/m}$ acts as a multiplier u_t of ${}^{J_i}\!/_{J_{i-1}}$ then for each representation $\lambda \in \Lambda_i$, $\lambda(u_t) \in \mathcal{B}(\mathcal{H}_i)$. The operator $\Delta_0^{\mathrm{i}t/m}$ thus preserves each fiber of the continuous field of C^* algebras defining ${}^{\Sigma_i}\!/_{\Sigma_{i-1}}$. We then need to show that its norm is bounded but being a multiplier of ${}^{J_i}\!/_{J_{i-1}}$ we have:

$$\sup_{\lambda \in \Lambda_i} \|\lambda(\Delta_0^{\mathrm{i}t/m})\|_{\mathcal{B}(\mathcal{H}_\lambda)} < +\infty$$

Therefore we have:

$$\Delta_0^{\mathrm{i}t/m} \in \mathscr{C}_b({}^{\Lambda_i}\!/_{\mathbb{R}_+^*}, \mathcal{K}(\mathcal{H}_i)) \subset \mathcal{M}\left({}^{\Sigma_i}\!/_{\Sigma_{i-1}}\right).$$

The crossed product ${}^{\Sigma_i}\!/_{\Sigma_{i-1}} \rtimes \mathbb{R}$ is therefore trivial and we have:

$${}^{\Sigma_i}\!/_{\Sigma_{i-1}} \rtimes \mathbb{R} \cong \mathscr{C}_0(\Lambda_i, \mathcal{K}(\mathcal{H}_i)).$$

The identification ${}^{\Lambda_i}\!/_{\mathbb{R}_+^*} \times \mathbb{R}_+^* \cong \Lambda_i$ is made by the choice of quasi-norm on G. ☐

We have thus far showed that the maps:

$$\Phi\colon {}^{\Sigma_i}\!/_{\Sigma_{i-1}} \rtimes \mathbb{R} \to {}^{J_i}\!/_{J_{i-1}},$$

were isomorphisms of C^*-algebras. We can then use the respective exact sequences to show inductively that each

$$\Phi\colon \Sigma_i \rtimes \mathbb{R} \to J_i,$$

is an isomorphism (starting from $i = 0$). The result for $i = r$ then concludes the proof. ☐

Remark 3.2.11. Although we have shown that $\Delta_0^{\mathrm{i}t/m}$ was a multiplier of each ${}^{\Sigma_i}\!/_{\Sigma_{i-1}}$ this does not mean that it is a multiplier of Σ_i. This would mean that the crossed product is trivial which only seem to hold in the commutative case.

As a corollary of the proof we obtain a decomposition result for the symbol algebra:

Theorem 3.2.12. *Let G be a graded Lie group*[4]*, $\emptyset = V_0 \subset V_1 \subset \cdots \subset V_r = \hat{G} \setminus \{1\}$ the Pedersen stratification of its unitary dual. Denote by $\Sigma_i(G)$ the ideal corresponding to the subspace ${V_i}/{\mathbb{R}_+^*}$ of the spectrum $\widehat{\Sigma(G)}$. This gives a nested sequence of ideals:*

$$\{0\} = \Sigma_0(G) \lhd \Sigma_1(G) \lhd \cdots \lhd \Sigma_r(G) = \Sigma(G)$$

and we have isomorphims:

$$\Sigma_i(G)/\Sigma_{i-1}(G) \cong \mathscr{C}_0\left(\Lambda_i/\mathbb{R}_+^*, \mathcal{K}_i\right).$$

Here $\Lambda_i = V_i \setminus V_{i-1}$, $\Lambda_i/\mathbb{R}_+^*$ is a locally compact Haussdorff space and $\mathcal{K}_i$ is the algebra of compact operators on a separable Hilbert space (of dimension 1 if $i = r$ and infinite dimensional otherwise).

This is a generalization of Epstein and Melrose decomposition of the symbol algebra in the contact case [34]. They showed that if (M, H) is a (compact) contact manifold then there is an exact sequence:

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow \Sigma(T_H M) \longrightarrow \mathscr{C}(\mathbb{S}H^*) \longrightarrow 0.$$

Regarding our result this corresponds to the Pedersen stratification of the Heisenberg group associated to H_x for each $x \in M$. Recall that this consists of $\Lambda_1 = \mathbb{R}^*$ and $\Lambda_2 = H_x^*$ and the $\mathbb{R}_+^*$-action on each component is a regular dilation. In this case we can extend the decomposition to the whole algebra (rather than the fibers over each point of M) because $T_H M$ is locally trivial (and our result clearly extends to locally trivial bundles of graded Lie groups). In general it is not clear if such a decomposition is possible since even the number of strata in Pedersen's stratification might vary from point to point. The examples given by the abelian case and the Heisenberg groups might also be misleading on one point. The strata $\Lambda_i/\mathbb{R}_+^*$ might not be compact, as shown in the following example.

Example 3.2.13. *Consider the Engel group whose Lie algebra $\mathfrak{g}$ is generated by X, Y, Z, T with $[X, Y] = Z, [X, Z] = T$ and the other brackets being zero. This is a 3-step nilpotent Lie group. Its representation theory can be computed through Kirillov's theory [53] and its Pedersen stratification gives for the symbol algebra:*

$$\Lambda_1/\mathbb{R}_+^* \cong \mathbb{R} \sqcup \mathbb{R}; \Lambda_2/\mathbb{R}_+^* \cong \{pt\} \sqcup \{pt\}; \Lambda_3/\mathbb{R}_+^* \cong \mathbb{S}^1.$$

The first strata is therefore non-compact.

[4] Or a locally trivial bundle of graded Lie groups.

3.3 Isomorphism for operators

3.3.1 The ideal J of Debord and Skandalis

In this section we introduce the analog of the ideal $\mathcal{J}(G)$ of Debord and Skandalis (see [30]) for $G = M \times M$ (although the idea directly generalizes to groupoids with filtered algebroid). Roughly speaking this ideal of $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ corresponds, in the commutative case, to the functions that vanish at infinite order on the zero section of TM seen as the zero fiber. In the noncommutative case we need to replace evaluation at zero by the trivial representation of the fiber. Elements of $\mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$ already vanish with infinite order on $T_H M$ so they will satisfy this condition. Therefore we need to declare which functions of $\mathscr{S}(\mathbb{V})$ will be in the ideal for an exponential chart $\exp^{\nabla,\psi} \colon \mathbb{U} \xrightarrow{\sim} \mathbb{V}$. Using the exponential map we can define the functions on the ideal working on $\mathbb{U}$. This allows us to use the Fourier transform $\mathcal{F} \colon \mathscr{S}(\mathfrak{t}_H M \times \mathbb{R}, \Omega^{1/2}) \xrightarrow{\sim} \mathscr{S}(\mathfrak{t}_H M^* \times \mathbb{R})$. Now the trivial representation of a fiber corresponds to the evaluation at 0 for the Fourier transform[5]. Consequently we can reformulate vanishing on the trivial representation by the vanishing of the Fourier transform at zero.

Definition 3.3.1. Let $\exp^{\nabla,\psi} \colon \mathbb{U} \xrightarrow{\sim} \mathbb{V}$ be an exponential chart for $\mathbb{T}_H M$. A function $f \in \mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ written as $f = f_0 + f_1 \in \mathscr{S}(\mathbb{V}) + \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$ is in the subspace $\mathcal{J}_H(M)$ if $\mathcal{F}\left(f \circ \exp^{\nabla,\psi} \circ \Psi\right)$ vanishes at infinite order on the zero section of $\mathfrak{t}_H M^* \times \mathbb{R}$.

Proposition 3.3.2. *A function $f = (f_t)_{t\in\mathbb{R}} \in \mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ is in $\mathcal{J}_H(M)$ if and only if for every $g \in \mathscr{C}_c^\infty(M \times M, \Omega^{1/2})$ the function $t \mapsto f_t * g$ extends to a function that belongs to $\mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$.*

Proof. Since $\mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$ is an ideal we can assume that f lies in an exponential chart $\exp^{\nabla,\psi} \circ \Psi \colon \mathbb{U} \xrightarrow{\sim} \mathbb{V}$. Writing the Taylor series at 0 of $t \mapsto \mathcal{F}\left(f_t \circ \exp^{\nabla,\psi} \circ \Psi\right)(x, {}^t\mathrm{d}\delta_t \eta)$ we see that every $t \mapsto f_t * g$ vanishes at order k at $t = 0$ if and only if the Taylor coefficients up to order k vanish as well. $\square$

Remark 3.3.3. The proof actually show that the functions $t \mapsto f_t * g$ have the same order of annulation as the one of the Fourier transform of f in local charts.

[5] Here we use the scalar Fourier transform. In the non-commutative setup a representation valued Fourier transform is often more suitable, see [61, 36]. However, since we are only interested in the trivial representation (which corresponds to a single point through the orbit method) the scalar valued transform is more manageable.

Corollary 3.3.3.1. *The set $\mathcal{J}_H(M)$ is a *-ideal of $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$.*

Proof. Directly follows from the previous proposition and the fact that the subalgebra $\mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$ is a *-ideal of $\mathscr{S}(\mathbb{T}_H M, \Omega^{1/2})$ itself. $\square$

3.3.2 Pseudodifferential operators as integrals

We now adapt the main result of [30] to the non-commutative case. To prove that a certain distribution on $M \times M$ is a pseudodifferential operator we will want to find a quasi-homogeneous extension to $\mathbb{T}_H M$. This is not very natural however and the natural extension is exactly homogeneous. We thus need a global version of Taylor's lemma 3.2.2. Its proof is similar to the previous one that can be found in [76], this global version can be found in [2] (proposition 3.4).

Lemma 3.3.4 (Global Taylor's lemma)**.** *Let $\exp^{\nabla,\psi} \colon \mathbb{U} \xrightarrow{\sim} \mathbb{V} \subset \mathbb{T}_H M$ be an exponential neighborhood. Let $\mathbb{P} \in \Psi_H^m(M)$ with support in $\mathbb{V}$. There is a (unique) smooth function $v \in \mathscr{C}^\infty((\mathfrak{t}_H M \times \mathbb{R}) \setminus (M \times \{(0,0)\}))$ such that:*

i) v is homogeneous of degree m: $\forall \lambda > 0, v(x, {}^t\mathrm{d}\delta_\lambda(\eta), t\lambda) = \lambda^m v(x, \eta, t)$.

ii) If $\chi \in \mathscr{C}_c^\infty(\mathfrak{t}_H M^* \times \mathbb{R})$ equal to 1 in the neighborhood of the zero section, then the map:

$$(x, \xi, t) \mapsto \exp^{\nabla,\psi *} \mathbb{P}(x, \xi, t) - \int_{\eta \in \mathfrak{t}_{H,x} M^*} e^{\mathrm{i}\eta(\xi)} (1 - \chi)(x, \eta, t) v(x, \eta, t)$$

is a smooth section of $\Omega^1(\mathfrak{t}_H M)$ over $\mathfrak{t}_H M \times \mathbb{R}$. It has support included in $\mathfrak{t}_H M \times [-T; T]$ for some $T > 0$, and all the derivatives w.r.t x and t are Schwartz in the ξ direction uniformly in x and t.

Conversely if a function v satisfies i) then one can find $\mathbb{P} \in \Psi_H^m(M)$ such that ii) is satisfied.

Theorem 3.3.5. *Let $m \in \mathbb{C}$, $f \in \mathcal{J}_H(M)$. The convolution operator by $\displaystyle\int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t}$ on $M \times M$ corresponds to a pseudodifferential operator of order m in the filtered calculus. Its principal symbol is given as an element of $\mathcal{K}^m(T_H M)$ by $\displaystyle\int_0^{+\infty} t^{-m} \delta_{t*} f_0 \frac{\mathrm{d}t}{t}$.*

Proof. If $f \in \mathscr{S}(\mathbb{R}^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$ then the integral converges absolutely. Proposition 3.3.2 then gives meaning to the integral $\displaystyle\int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t}$ as a convolution operator. Moreover we can assume that f is supported in an exponential chart and work on $\mathfrak{t}_H M \times \mathbb{R}$. Using the converse of Taylor's Lemma we need to find a homogeneous extension of the Fourier transform of the operator on $\mathbb{U}$ where $\mathbb{U} \xrightarrow{\sim} \mathbb{V}$ is the exponential neighborhood where f is supported. We can rewrite the operator as

$$\int_0^{+\infty} s^{-m} \alpha_{s*} f_1 \frac{\mathrm{d}s}{s},$$

and thus consider the natural extension:

$$\mathbb{P}_t := \int_0^{+\infty} s^{-m} \alpha_{s*} f_t \frac{\mathrm{d}s}{s}.$$

This extension is exactly homogeneous of degree m for the push-forward by the zoom action. Since α restricts to δ on the zero fiber we obtain that $\mathbb{P}_0 = \int_0^{+\infty} s^{-m} \delta_{s*} f_0 \frac{\mathrm{d}s}{s}$. In order to use Taylor's lemma we consider f as a function in $\mathscr{S}(\mathbb{U})$, we now need to show that the Fourier transform of $\mathbb{P}$ defines a smooth function. We have $f_t \in \mathscr{S}(\mathfrak{t}_H M)$ so its Fourier transform is well defined. Recall that under the exponential map, the zoom action on $\mathfrak{t}_H M \times \mathbb{R}$ becomes β with:

$$\beta_\lambda(x, \xi, t) = (x, \mathrm{d}\delta_\lambda(\xi), \lambda^{-1} t).$$

The action on $\mathfrak{t}_H M^* \times \mathbb{R}$ is then ${}^t\beta$ where:

$${}^t\beta_\lambda(x, \eta, t) = (x, {}^t\mathrm{d}\delta_\lambda(\eta), \lambda t).$$

If $\mathcal{F} \colon \mathscr{S}(\mathfrak{t}_H M \times \mathbb{R}) \to \mathscr{S}(\mathfrak{t}_H M^* \times \mathbb{R})$ denotes the Fourier transform in the $\mathfrak{t}_H M$ directions then we have:

$$\mathcal{F} \circ \beta_{\lambda*} = {}^t\beta_\lambda^* \circ \mathcal{F}.$$

This result yields:

$$\hat{\mathbb{P}}(x, \eta, t) = \int_0^{+\infty} s^{-m} \hat{f}_{ts}(x, {}^t\mathrm{d}\delta_s(\eta)) \frac{\mathrm{d}s}{s}.$$

Now $s \mapsto \hat{f}_{ts}(x, {}^t\mathrm{d}\delta_s(\eta))$ is in $\mathscr{S}(\mathbb{R}_+^*)$ by assumption thus the integral converges (we can factor out a high enough power of s to compensate for the $s^{-(m+1)}$ and get the convergence at 0). This convergence is uniform for (x, η, t) in any compact subset of $\mathfrak{t}_H M^* \times \mathbb{R} \setminus (M \times \{0\})$. Therefore the function $\hat{\mathbb{P}}$ is smooth on $\mathfrak{t}_H M^* \times \mathbb{R} \setminus (M \times \{0\})$ and homogeneous by construction. Therefore the operator given by Taylor's lemma gives a quasi homogeneous extension to $\mathbb{P}_1$ (it is equal to $\mathbb{P}_1$ at $t = 1$ modulo a smoothing operator). □

Remark 3.3.6. We actually see that to obtain the convergence of the integral we only need $\hat{f}$ to vanish on the zero section at order superior or equal to $\Re(m)+1$. This will be useful later when using the factorisation lemma.

Lemma 3.3.7. *Let $f_0 \in \mathscr{S}(T_H M)$ such that $\hat{f}_0$ vanishes on $M \times \{0\}$ at order $k \in \mathbb{N} \cup \{+\infty\}$, then there exists $f \in \mathscr{S}(\mathbb{T}_H M)$ such that, in an exponential chart, $(\hat{f}_t)_{t\in\mathbb{R}}$ vanishes on $M \times \{0\} \subset \mathfrak{t}_H M \times \mathbb{R}$ at order k and $f_0 = f(\cdot, 0)$.*

Proof. Let us consider a function $\varphi \in \mathscr{C}_c^\infty(\mathfrak{t}_H M)$ such that $\varphi(x,0) = 1$ and $\lim\limits_{t\to+\infty} \widehat{\varphi} \circ {}^t\mathrm{d}\delta_t = \delta_M$ where δ_M is the distribution corresponding to integrating along the zero section of $\mathfrak{t}_H M$ (it corresponds to the Dirac distribution at zero on each fiber of $\mathfrak{t}_H M$). Then we set:

$$\tilde{f}(x,\xi,t) = f_0(x,\xi)\varphi(x, \mathrm{d}\delta_t(\xi)).$$

We have $\widehat{\tilde{f}}_t(x,\eta) = \widehat{f_0}(x,\eta) *_{ab} \varphi(x, {}^t\mathrm{d}\delta_{t^{-1}}(\eta))$ where $*_{ab}$ is the convolution product on $\mathfrak{t}_H^* M$ seen as a bundle of abelian groups and we obtain the vanishing of $\widehat{\tilde{f}}$ at the right order. However the function $\tilde{f}$ is not in the Schwartz class as $\tilde{f}(x,0,t) = f_0(x,0)$ is constant in t hence not rapidly decreasing in general. This can be corrected by multiplying $\tilde{f}$ by a cutoff function $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ that is constant equal to 1 in a neighborhood of 0. □

Proposition 3.3.8. *Every pseudodifferential operator can be written as in theorem 3.3.5: if $P \in \Psi_H^m(M)$ there exists $f \in \mathcal{J}_H(M)$ such that*

$$P = \int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t}.$$

Proof. The result is trivial if $P \in \mathscr{C}_c^\infty(M \times M, \Omega^{1/2})$. If P is a pseudodifferential operator of order m then let $\sigma^m(P) \in \Sigma^m(T_H M)$ be its principal symbol, its associated full symbol $v \in \mathscr{C}^\infty(\mathfrak{t}_H M^* \setminus (M \times \{0\})$ obtained by 3.3.4 is homogeneous of order m and can thus be written as $\displaystyle\int_0^{+\infty} t^{-m}\delta_t^* \hat{f}_0 \frac{\mathrm{d}t}{t}$ for a function $f_0 \in \mathscr{S}_0(\mathbb{T}_H M, \Omega^{1/2})$. We then use lemma 3.3.7 to extend f_0 to a function $f \in \mathcal{J}_H(M)$. Then P and $P_f := \displaystyle\int_0^{+\infty} t^{-m} f_t \frac{\mathrm{d}t}{t}$ have the same principal symbol and thus belong to $\Psi_H^{m-1}(M)$. We then iterate the argument and approach P with an asymptotic series[6]. □

Lemma 3.3.9. *Let $f, g \in \mathcal{J}_H(M)$ then $u \mapsto (f_t * g_{tu})_{t\in\mathbb{R}} \in \mathscr{S}(\mathbb{R}_+^*, \mathcal{J}_H(M))$.*

[6] Recall from [81] that the pseudodifferential filtered calculus is asymptotically complete.

Proof. This is the same type of reasoning as in 3.3.2 but applied to the groupoid $\mathbb{T}_H M$ instead of $M \times M$ (recall our contruction is valid for an arbitraty groupoid). Here $\mathbb{T}_H M$ has its algebroid naturally filtered by the filtration on TM and on $\mathfrak{t}_H M$ (the j-th stratum of $\mathfrak{t}_H M$ is $\bigoplus_{i=1}^{j} H^i/_{H^{i-1}}$). □

Proposition 3.3.10. *Let $f = (f_t)_{t\in\mathbb{R}} \in \mathcal{J}_H(M)$ and $P \in \Psi_H^s(M)$ with $\Re(s) \leq 0$, then $(f_t * P)_{t\in\mathbb{R}^*}$ extends to an element of $\mathcal{J}_H(G)$. Moreover its value at $t = 0$ is $f_0 * \sigma^s(P)$ with $\sigma^s(P) \in \mathcal{K}^s(T_H M)$ seen as a multiplier of $\mathscr{S}_0(T_H M)$.*

Proof. Write P as an integral $\int_0^{+\infty} g_t \frac{\mathrm{d}t}{t}$ then use the previous lemma with f and g. □

3.3.3 Completion and crossed product of the pseudodifferential algebra

We now consider the C^*-completion of the previous algebras. From now on we restrict the tangent groupoid over $\mathbb{R}_+$ and denote it by

$$\mathbb{T}_H^+ M := \mathbb{T}_H M_{|M\times\mathbb{R}_+}.$$

We also denote by $\mathcal{J}_H^+(M)$ the space of restriction of functions in $\mathcal{J}_H(M)$ to $\mathbb{T}_H^+ M$. The respective completions of the algebras $\mathscr{S}(\mathbb{T}_H^+ M, \Omega^{1/2})$, $\mathcal{J}_H^+(M)$ and $\mathscr{S}(\mathbb{R}_+^*, \mathscr{C}_c^\infty(M\times M, \Omega^{1/2})$ are the C^*-algebras $C^*(\mathbb{T}_H^+ M), C_0^*(\mathbb{T}_H^+ M)$ and $\mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M)))$. They sit in the obvious exact sequences[7]

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M))) \longrightarrow C_{(0)}^*(\mathbb{T}_H^+ M) \longrightarrow C_{(0)}^*(T_H M) \longrightarrow 0.$$

We also denote by $\Psi_H^*(M)$ the C^*-completion of $\Psi_H^0(M)$.

Proposition 3.3.11. *Let $P \in \Psi_H^0(M)$, the action of on $\mathcal{J}_H^+(M)$ defined in proposition 3.3.10 extends to a multiplier of $C_0^*(\mathbb{T}_H M^+)$. Moreover the resulting morphism $\Psi_H^0(M) \to \mathcal{M}(C_0^*(\mathbb{T}_H M^+))$ is continuous and if moreover $P \in \Psi_H^{-1}(M)$, then it preserves the ideal $\mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M)))$.*

Proof. Recall that $C_0^*(\mathbb{T}_H^+ M)$ is a continuous field of C^*-algebras over $\mathbb{R}_+$ with fiber at $t \neq 0$ equal to $\mathcal{K}(L^2(M)) = C^*(M \times M)$ and its fiber at $t = 0$ is $C_0^*(T_H M)$. We thus have for each $t \neq 0$, $\|(P * f)_t\| = \|P\|_{\Psi^*}\|f_t\|$ and $\|(P * f)_0\| \leq \|\sigma(P)\|\|f_0\| \leq \|P\|_{\Psi^*}$. Here $\|\cdot\|_{\Psi^*}$ denotes the norm of $\Psi_H^*(M)$,

[7] The (0) means that the sequences are exact with or without the 0.

i.e. the norm of the operator $P*$ acting on $L^2(M)$. The last inequality follows from the continuity of the principal symbol map. From these inequalities it follows that:

$$\forall f \in \mathcal{J}_H^+(M), \|P * f\| \leq \|P\|_{\Psi^*}\|f\|.$$

Therefore the operator $(f_t)_{t\geq 0} \mapsto (P * f_t)_{t\geq 0}$ extends continuously to an operator on $C_0^*(\mathbb{T}_H M)$. The morphism $\Psi_H^0(M) \to \mathcal{M}(C_0^*(\mathbb{T}_H M))$ is continuous and extends to

$$\Psi_H^*(M) \to \mathcal{M}(C_0^*(\mathbb{T}_H M)).$$

For the last statement it follows from the fact that the restriction of $P * f$ to the zero fiber is the action of the principal symbol which is zero on the subspace $\Psi_H^{-1}(M)$. □

Remark 3.3.12. More precisely if $P \in \Psi_H^{-1}(M)$ and $f \in \mathcal{J}_H^+(M)$ then one can prove that $P * f \in \mathscr{S}(\mathbb{R}_+^*, \mathscr{C}_c^\infty(M \times M, \Omega^{1/2}))$.

Let $\Delta \in \Psi_H^q(M)$ be a positive Rockland differential operator (for some even $q > 0$). Using theorem 2.2.37 we have a continuous family of operators $\Delta^{\mathrm{i}t/q} \in \Psi_H^{\mathrm{i}t}(M)$. These operators act on $\mathcal{J}_H^+(M)$ as the order zero operators. Analogously to the previous proposition, this action extends to unitary multipliers of $C_0^*(\mathbb{T}_H M^+)$. The operators also give an $\mathbb{R}$-action on $\Psi_H^*(M)$ with $\mathrm{Ad}(\Delta^{\mathrm{i}t/q})$. We thus get a *-homomorphism:

$$\Phi\colon \Psi_H^*(M) \rtimes \mathbb{R} \to C_0^*(\mathbb{T}_H^+ M).$$

Theorem 3.3.13. *The morphism Φ preserves the respective exact sequences of $\Psi_H^*(M)$ and $C_0^*(\mathbb{T}_H M^+)$, i.e. there is a commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M))) & \longrightarrow & \Psi_H^*(M) \rtimes \mathbb{R} & \longrightarrow & \Sigma(T_H M) \rtimes \mathbb{R} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi_0} & & \\
0 & \longrightarrow & \mathscr{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M))) & \longrightarrow & C_0^*(\mathbb{T}_H^+ M) & \longrightarrow & C_0^*(T_H M) & \longrightarrow & 0.
\end{array}$$

Here $\Phi_0\colon \Sigma(G) \rtimes \mathbb{R} \to C_0^*(T_H M)$ is the isomorphism of the previous section. In particular Φ is an isomorphism.

Proof. This is a direct consequence of the previous proposition. The fact that the crossed product is trivial on the ideal in the first row is because each $\Delta^{\mathrm{i}t/q}$ extends to a multiplier of $\mathcal{K}(L^2(M)) = \overline{\Psi_H^{-1}(M)}$. The fact that Φ_0 is the isomorphism of the previous section is deduced from proposition 3.3.10. The fact that Φ is an isomorphism is then short five lemma. □

From this result we can state another one similar to Debord and Skandalis original approach, also obtained in the filtered case by Ewert in [35]:

Corollary 3.3.13.1. *There is a canonical bimodule yielding a Morita equivalence between the algebras $\Psi^*_H(M)$ and $C^*_0(\mathbb{T}_H M) \rtimes_\alpha \mathbb{R}^*_+$ preserving their respective exact sequences.*

Proof. The zoom action α is dual to the one defined with $(\Delta^{\mathrm{i}t/q})_{t\in\mathbb{R}}$ thus this is just an application of Takai duality. □

This result allows for some index theoretic consequences[8]:

Proposition 3.3.14. *Let (M,H) be a (compact) filtered manifold then the algebras of symbols $\Sigma(T_H M)$ and $\Sigma(TM)$ are KK-equivalent. The pseudodifferential algebras $\Psi^*_H(M)$ and $\Psi^*(M)$ are KK-equivalent as well.*

Proof. We use the last corollary to get a Morita equivalence

$$\Sigma(T_H M) \sim_{mor} C^*_0(T_H M) \rtimes \mathbb{R}^*_+.$$

The same goes in the commutative case with $\Sigma(TM) \sim_{mor} C^*_0(TM) \rtimes \mathbb{R}^*_+$ (which is just the isomorphism $\mathscr{C}_0(\mathbb{S}^* M) \cong \mathscr{C}_0(T^*M \setminus 0) \rtimes \mathbb{R}^*_+$). Now we can use the Connes-Thom isomorphism for both groups bundles $T_H M$ and TM to get a KK-equivalence $C^*(TM) \sim_{KK} C^*(T_H M)$ which restricts to the kernels of trivial representations in a $\mathbb{R}^*_+$-equivariant way, we thus get $C^*_0(TM) \sim_{KK^{\mathbb{R}^*_+}} C^*_0(T_H M)$ and thus the desired KK-equivalence by composition (remember a Morita equivalence induces a KK-equivalence).

For the pseudodifferential algebras we also use the Morita equivalence $\Psi^*_H(M) \sim_{mor} C^*_0(\mathbb{T}^+_H M)$. We now need to show a KK-equivalence:

$$C^*_0(\mathbb{T}^+_H M) \sim_{KK} C^*_0(\mathbb{T}^+ M).$$

Using the (classical) adiabatic groupoid of $\mathbb{T}_H M$ [9] and restricting it over $[0;1]^2$, we obtain a groupoid $\mathbb{G} \rightrightarrows M \times [0;1]^2$. It has the following restrictions with coordinates $(s,t) \in [0;1]^2$:

- the restriction to a fiber for $s \neq 0$ of $\mathbb{G}$ is $\mathbb{T}_H M_{|[0;1]}$
- the restriction to the fiber $s = 0$ of $\mathbb{G}$ is $TM \times [0;1]$ (after a choice of splitting for $\mathfrak{t}_H M$).
- the restriction to a fiber for $t \neq 0$ of $\mathbb{G}$ is $\mathbb{T}M_{|[0;1]}$
- the restriction to the fiber $t = 0$ of $\mathbb{G}$ is $T_H M^{ad}_{|[0;1]}$

[8] The first KK-equivalence is also in [35].

[9] One could also use the filtered adiabatic groupoid of $\mathbb{T}M$ for the filtration given by the $H^i \times \mathbb{R}$. This results in an isomorphic groupoid under the flip of coordinates $(s,t) \mapsto (t,s)$ on $\mathbb{R}^2$.

Now using the KK-equivalences obtained by the exact sequences induces by the row and columns of the square we get the KK-equivalences:

$$\begin{aligned} C^*(\mathbb{T}_H M_{|[0;1]}) &\sim C^*(TM \times [0;1]) \\ &\sim C^*(TM) \\ &\sim C^*(T_H M^{ad}_{|[0;1]}) \\ &\sim C^*(\mathbb{T}M_{|[0;1]}) \end{aligned}$$

Now this KK-equivalence commutes with the evaluation maps on the same fiber and thus restricts to a KK-equivalence between the groupoid algebras $C^*(\mathbb{T}_H M_{|[0;1)})$ and $C^*(\mathbb{T}M_{|[0;1)})$. We can now use a homeomorphism between $[0;1)$ and $\mathbb{R}_+$ to get a KK-equivalence $C^*(\mathbb{T}_H^+ M) \sim C^*(\mathbb{T}^+ M)$. By construction this equivalence restricts to a KK-equivalence between $C_0^*(\mathbb{T}_H^+ M)$ and $C_0^*(\mathbb{T}^+ M)$ [10] in an $\mathbb{R}_+^*$ equivariant way. We thus obtain the KK-equivalence between their crossed products $C_0^*(\mathbb{T}_H^+ M) \rtimes \mathbb{R}_+^*$ and $C_0^*(\mathbb{T}^+ M) \rtimes \mathbb{R}_+^*$ and thus between $\Psi_H^*(M)$ and $\Psi^*(M)$.

□

Roughly speaking, the last proposition shows that one could not hope to obtain further index theoretic invariants from the filtered calculus, other than the ones that could previously be obtained from the classical pseudodifferential calculus. This however does not help when one wants to compute the index of an actual operator is to invert the Connes-Thom isomorphism that was used here (see [7, 64]).

[10] The same exact sequences used to obtain the KK-equivalences can be written for the C_0^* algebras.

Chapter 4

Transversal index theory

In this chapter we take the setup of a filtered manifold (M, H) and add an integrable subbundle $H^0 \subset H^1$. We define a notion of transverse symbols to the foliation induced by H^0, a "restriction in the transverse direction" map for the symbols and a transversal Rockland condition in the framework of filtered calculus. We show the existence of a transverse index for symbols satisfying this condition and a Poincare duality type result between the class of an operator and the one of its symbol. In the philosophy of noncommutative geometry this setup corresponds to a filtration on the space of leaves of the foliation so we will pay a special attention to the case of foliations given by fibrations. The chapter is structured as follows: the first part is devoted to the geometric framework, the holonomy actions and the introduction of the different groupoids that will be used in the subsequent parts. The second part details the restriction of symbols to the transverse part in the setting of filtered calculus which allows to define a "transversally Rockland" condition. Finally we give two constructions of the transverse cycle, one from the symbol and using KK-theoretic tools and the other one more analytic, similar to the one in [44] using a pseudodifferential operator instead of its symbol and we prove the equality between the two KK-classes, yielding a Poincare duality type result. We also carry out the study of two main examples, namely the foliations given by fibrations and the ones arising from discrete group actions.

4.1 Geometric preliminaries

4.1.1 The setting and the holonomy action

Let M be a manifold. A foliated filtration is a filtration of the tangent bundle $H^0 \subset H^1 \subset \cdots \subset H^r = TM$ such that

$$\forall i, j \geq 0, \left[\Gamma(H^i), \Gamma(H^j)\right] \subset \Gamma(H^{i+j})$$

. This condition directly implies that H^0 is integrable and that

$$H^1 \subset \cdots \subset H^r = TM,$$

is a Lie filtration on M as described in Chapter 2. Moreover, the Lie bracket of vector fields then gives maps:

$$\Gamma\left(H^i / H^{i-1}\right) \times \Gamma\left(H^j / H^{j-1}\right) \to \Gamma\left(H^{i+j} / H^{i+j-1}\right),\ i, j \geq 0,$$

so that

$$\mathfrak{t}_H M = H^1 \oplus H^2 / H^1 \oplus \cdots \oplus TM / H^{r-1}$$

and

$$\mathfrak{t}_{H/H^0} M = H^1 / H^0 \oplus H^2 / H^1 \oplus \cdots \oplus TM / H^{r-1}$$

are both endowed with a Lie algebroid structure over M (with trivial anchor). In fact for $i, j \geq 1$ the bracket is $\mathscr{C}^\infty(M)$-bilinear and hence gives a structure of (nilpotent) Lie algebra bundles to $\mathfrak{t}_H M$ and $\mathfrak{t}_{H/H^0} M$ (see [80, 18, 63] for more details on this construction). Actually, for fixed $x \in M$, $H^0_x \subset \mathfrak{t}_{H,x} M$ is an ideal and the quotient Lie algebra is $\mathfrak{t}_{H/H^0,x} M$.

Using Baker-Campbell-Hausdorff formula we integrate these Lie algebra bundles into nilpotent Lie group bundles $T_H M$ and $T_{H/H^0} M$ and the quotient map $\mathfrak{t}_H M \to \mathfrak{t}_{H/H^0} M$ gives a quotient map $T_H M \to T_{H/H^0} M$.
Another groupoid useful to our study is the holonomy groupoid of the foliation given by H^0, let us denote it $\mathrm{Hol}(H^0)$. We want to show that this groupoid acts on $T_{H/H^0} M$, preserving its groupoid structure (i.e. for all $\gamma \in \mathrm{Hol}(H^0), \gamma \colon T_{H/H^0,s(\gamma)} M \to T_{H/H^0,r(\gamma)} M$ is a group homomorphism). In order to do so we need the following result :

Lemma 4.1.1. *Let $G \rightrightarrows M$ be a Lie groupoid and $Y \subset M$ a submanifold. Let ρ be the anchor map of the associated algebroid $\mathcal{A}G \to M$. If for all $y \in Y$, $\rho(\mathcal{A}_y G) \subset T_y Y$ then $G \cdot Y \subset Y$, i.e. $\forall \gamma \in G, s(\gamma) \in Y \Leftrightarrow r(\gamma) \in Y$.*

Proof. $\rho(\mathcal{A}_y G)$ is the tangent space to the G-orbit passing through y hence the condition implies that an orbit passing through Y stays in Y hence the result holds true. □

Corollary 4.1.1.1. *Let $F \subset TM$ be a foliation and $H \subset TM$ be an arbitrary subbundle. If $[\Gamma(F), \Gamma(H)] \subset \Gamma(H)$ then ${}^{H}\!/\!{}_{F}$ is invariant under the action $\mathrm{Hol}(F) \curvearrowright {}^{TM}\!/\!{}_{F}$.*

Proof. We apply the lemma to the groupoid $G = \mathrm{Hol}(F) \ltimes {}^{TM}\!/\!{}_{F} \rightrightarrows {}^{TM}\!/\!{}_{F}$ with $Y = {}^{H}\!/\!{}_{F}$. Let $x \in M$, $\xi \in H_x$. Denote by $\bar{\xi}$ the class of ξ modulo F_x. We have the exact sequence:

$$0 \longrightarrow {}^{TM}\!/\!{}_{F} \longrightarrow T\left({}^{TM}\!/\!{}_{F}\right) \xrightarrow{\mathrm{d}\pi} TM \longrightarrow 0\ ,$$

where the first copy of ${}^{TM}\!/\!{}_{F}$ corresponds to vertical vector fields for the bundle $\pi\colon {}^{TM}\!/\!{}_{F} \to M$. When restricted to Y we obtain

$$0 \longrightarrow {}^{H}\!/\!{}_{F} \longrightarrow TY \xrightarrow{\mathrm{d}\pi} TM \longrightarrow 0\ .$$

The direct image of $\mathcal{A}_{(x,\bar{\xi})}G$ by the anchor map is then

$$0 \longrightarrow [F, \bar{\xi}] \longrightarrow \rho(\mathcal{A}_{(x,\bar{\xi})}G) \xrightarrow{\mathrm{d}\pi} F_x \longrightarrow 0\ ,$$

where $[F, \bar{\xi}] = \{\overline{[X,Y]}(x), X \in \Gamma(F), Y \in \Gamma(TM)\bar{Y}(x) = \bar{\xi}\} \subset {}^{H_x}\!/\!{}_{F_x}$ because $[\Gamma(F), \Gamma(H)] \subset \Gamma(H)$. We are thus under the conditions of the previous lemma and ${}^{H}\!/\!{}_{F}$ is $\mathrm{Hol}(F)$-invariant. □

The bracket conditions on foliated Lie filtrations combined with the corollary gives that $\mathrm{Hol}(H^0)$ acts on each ${}^{H^i}\!/\!{}_{H^0}$ hence on each ${}^{H^i}\!/\!{}_{H^{i-1}}$ $(i \geq 0)$. The action is compatible with the brackets

$$\Gamma({}^{H^i}\!/\!{}_{H^{i-1}}) \times \Gamma({}^{H^j}\!/\!{}_{H^{j-1}}) \to \Gamma({}^{H^{i+j}}\!/\!{}_{H^{i+j-1}}).$$

This gives an action $\mathrm{Hol}(H^0) \curvearrowright \mathfrak{t}_{H/H^0}M$ compatible with the algebroid structure on $\mathfrak{t}_{H/H^0}M$.

Corollary 4.1.1.2. *The action $\mathrm{Hol}(H^0) \curvearrowright \mathfrak{t}_{H/H^0}M$ lifts to a groupoid action $\mathrm{Hol}(H^0) \curvearrowright T_{H/H^0}M$ preserving the groupoid structure on $T_{H/H^0}M$.*

Proof. Since $T_{H/H^0}M$ is the s-simply connected groupoid integrating $\mathfrak{t}_{H/H^0}M$, we can lift the action of $\mathrm{Hol}(H^0)$ to a groupoid preserving action. □

Note that $\mathrm{Hol}(H^0)$ only acts on the normal bundle ${}^{TM}\!/\!{}_{H^0}$ and not on the whole tangent bundle to M. Hence it does not act on $\mathfrak{t}_H M$ nor on $T_H M$.

Example 4.1.2. *Let $H^1 \subset \cdots \subset H^r = TM$ be a Lie filtration. Let G be a Lie group acting on M locally freely and preserving each H^i. Then by differentiation the foliation $\mathfrak{g} \ltimes M$ induced by the action (which is a regular foliation because the action is locally free) also preserves each H^i, hence $\tilde{H}^0 = \mathfrak{g} \ltimes M$ and $\tilde{H}^i = H^i + \mathfrak{g} \ltimes M$ defines a foliated filtration if we assume the $\tilde{H}^i$ to be subbundles.*

Remark 4.1.3. In general we cannot assume that $\mathfrak{g} \ltimes M \subset H^1$, for instance the Reeb foliation on a contact manifold has its leaves transverse to the contact distribution although and also preserves it. Moreover in our setting we have to add the assumptions that each $\tilde{H}^i$ is a subbundle. This condition could be removed if one worked with filtration of the Lie algebra of vector fields by $\mathscr{C}^\infty(M)$ sub-modules as in [2].

Example 4.1.4. *In the same fashion, let $H^1 \subset \cdots \subset H^r = TM$ be a Lie filtration and $\mathcal{F} \subset TM$ be an arbitrary foliation such that*

$$\forall i \geq 1, \left[\Gamma(\mathcal{F}), \Gamma(H^i)\right] \subset \Gamma(H^i)$$

then $\tilde{H}^0 = \mathcal{F}$ and $\tilde{H}^i = \mathcal{F} + H^i$ defines a foliated filtration if each $\tilde{H}^i$ is a subbundle.

Example 4.1.5. *Let $\pi \colon M \to B$ be a fibration (with connected fibers). Assume B is a filtered manifold, let $\bar{H}^1 \subset \cdots \subset \bar{H}^r = TB$ be the corresponding Lie filtration. Let $H^0 = \ker(\mathrm{d}\pi)$ be the subbundle corresponding to the foliation induces by the fibers of π. Let $H^i = \pi^* \bar{H}^i \subset TM$ for $i \geq 1$. Then $H^0 \subset H^1 \subset \cdots \subset H^r$ is a foliated filtration on M. Furthermore, π induces a group isomorphism $T_{H/H^0,x}M \cong T_{\bar{H},\pi(x)}B$ for every $x \in M$, i.e. $T_{H/H^0}M = \pi^* T_{\bar{H}}B$.*

Example 4.1.6. *Let $n \in \mathbb{N}$ and $\omega \in \Omega^1(M, \mathbb{R}^n)$ a (n-tuple of) differential form(s) of locally constant rank. Let $H^1 = \ker(\omega)$ and assume there exists subbundles $H^i \subset TM$ with $\Gamma(H^{i+1}) = \Gamma(H^i) + \left[\Gamma(H^i), \Gamma(H^1)\right]$, assume the existence of $r \geq 1$ such that $H^r = TM$ (we can relax this condition, see remark 4.2.18). Define also $H^0 = \ker(\mathrm{d}\omega_{|H^1})$. Then $H^0 \subset \cdots \subset H^r = TM$ is a foliated filtration.*
Indeed H^0 is a foliation : if $X, Y \in \Gamma(H^0)$ then since $H^0 \subset H^1$:

$$\omega([X,Y]) = X \cdot \omega(Y) - Y \cdot \omega(X) - \mathrm{d}\omega(X,Y) = 0$$

so $[X,Y] \in \Gamma(H^1)$. Moreover if $Z \in \Gamma(H^1)$ then the same computation shows that $[X,Z] \in \Gamma(H^1)$

$$\mathrm{d}\omega([X,Y],Z) = [X,Y] \cdot \omega(Z) - Z \cdot \omega([X,Y]) - \omega([[X,Y],Z]) = 0$$

since by Jacobi's identity : $[[X,Y],Z] = -[[Y,Z],X] - [[Z,X],Y] \in \Gamma(H^1)$. *Thus far we have shown that* H^0 *is a foliation and that it acts on* H^1*. The action on* $H^i, i \geq 2$ *is then easily proved by induction.*

4.1.2 Deformation groupoids

Following the construction of Connes' tangent groupoid (see [21]) and due to its interest in pseudodifferential calculus and index theory (see e.g. [30, 31, 32, 80]), a deformation groupoid taking into account the filtered structure of $T_H M$ was constructed in [18, 63, 80]. We do not recall the construction here but we still construct its algebroid through the module of sections:

$$\Gamma(\mathbb{t}_H M) = \{X \in \Gamma(TM \times \mathbb{R}) \ / \ X_{|t=0} = 0 \ \text{ and } \ \forall i \geq 1,\ \partial_t^i X_{|t=0} \in \Gamma(H^i)\}.$$

The algebroid on $M \times \mathbb{R}$ thus obtained is almost injective and hence integrable thanks to a theorem of Debord (see [29]) into an s-connected groupoid $\mathbb{T}_H M$ (which is minimal in some sense to ensure the unicity of such groupoid). The references listed above give explicit constructions of this deformation groupoid. The groupoid $T_{H/H^0}M$, on the other hand, does not have a "good" deformation groupoid as it is not directly constructed from a groupoid with filtered algebroid. We can however deform its crossed product with the holonomy groupoid. In order to do this, we give another description of the algebroid of $\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M$.

Let $\mathfrak{t}_H^{\mathrm{Hol}} M = H^0 \oplus {}^{H^1}\!/_{H^0} \oplus \cdots \oplus {}^{TM}\!/_{H^{r-1}}$ be the algebroid whose anchor is given by the injection $H^0 \to TM$ and bracket of sections given by the same kind of formula as for $\mathfrak{t}_H M$. We then have $\mathfrak{t}_H^{\mathrm{Hol}} M = H^0 \ltimes \mathfrak{t}_{H/H^0} M$ and because $\mathrm{Hol}(H^0)$ acts on $\mathfrak{t}_{H/H^0} M$, we can adapt Nistor's proof of integration of crossed product algebroids from [67] and integrate $\mathfrak{t}_H^{\mathrm{Hol}} M$ into $\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M$. Let us now define the deformation algebroid through its module of sections

$$\Gamma(\mathbb{t}_H^{\mathrm{Hol}} M) = \{X \in \Gamma(TM \times \mathbb{R}) \ / \ \forall i \geq 0,\ \partial_t^i X_{|t=0} \in \Gamma(H^i)\}.$$

It is a subalgebroid of the algebroid of vector fields on $M \times \mathbb{R}$, hence a singular foliation. This foliation is almost regular, the anchor being injective exept for $t = 0$. We can thus construct by Debord's result its holonomy groupoid $\mathbb{T}_H^{\mathrm{Hol}} M$. It is a smooth groupoid over $M \times \mathbb{R}$ that integrates $\mathbb{t}_H^{\mathrm{Hol}} M$ and that algebraically satisfies

$$\mathbb{T}_H^{\mathrm{Hol}} M = M \times M \times \mathbb{R}^* \sqcup \mathrm{Hol}(H^0) \ltimes T_{H/H^0}M \times \{0\}.$$

Another construction of $\mathbb{T}_H^{\mathrm{Hol}} M$ which details the smooth structure is given in [63], example 5.4.

4.2 Transverse cycles

4.2.1 The KK-cycle of the symbol

Restriction of the symbol

Let G_1, G_2 be arbitrary Lie groupoids with same objects and $\pi\colon G_1 \to G_2$ a surjective homomorphism (over the identity). Integration along the kernel of π gives a *-homomorphism $\int_{\ker(\pi)} : C^*(G_1) \to C^*(G_2)$ (here the groupoid C^*-algebra is the maximal one, the reduced can be used if $\ker(\pi)$ is amenable[1]). Here we use this fact with the morphism $T_H M \to T_{H/H^0} M$ and we obtain the surjective morphism of restriction to the transverse directions :

$$\int_{H^0} : C^*(T_H M) \to C^*(T_{H/H^0} M)$$

which, by surjectivity, also extends to the multiplier algebras

$$\int_{H^0} : \mathcal{M}(C^*(T_H M)) \to \mathcal{M}(C^*(T_{H/H^0} M)).$$

This morphism sends principal symbols of H-pseudodifferential operators to their restriction to the directions transverse to H^0.

Example 4.2.1. *If $H^1 = TM$ we have $T_H M = TM$ and $T_{H/H^0} M = {}^{TM}\!/_{H^0}$ and under Fourier transform we have isomorphisms $C^*(TM) \cong \mathscr{C}_0(T^*M)$ and*
$C^*\left({}^{TM}\!/_{H^0}\right) \cong \mathscr{C}_0((H^0)^\perp)$. We then have the commutative diagram

$$\begin{array}{ccc} C^*(TM) & \xrightarrow{\int_{H^0}} & C^*\left({}^{TM}\!/_{H^0}\right) \\ \Vert\wr & & \Vert\wr \\ \mathscr{C}_0(T^*M) & \longrightarrow & \mathscr{C}_0((H^0)^\perp) \end{array}$$

where the bottom arrow corresponds to the restriction of functions on T^*M to
$(H^0)^\perp \subset T^*M$. This example justifies the name of "restriction" for the morphism $\int_{H^0}$.

[1] The trivial representation of $\ker(\pi)$ is then weakly contained in the regular, by continuity of induction then the regular representation of G_2 is weakly contained in the regular representation of G_1 hence the continuity for the morphism between the reduced C^*-algebras

Using the functoriality of symbols and the integration on H^0 we get a commutative diagram

$$\begin{array}{ccc} \bar{S}^0_0(T_H M) & \longrightarrow & \bar{S}^0_0(T_{H/H^0}) \\ \downarrow & & \downarrow \\ \mathcal{M}(C^*(T_H M)) & \xrightarrow{\int_{H^0}} & \mathcal{M}(C^*(T_{H/H^0})) \end{array}$$

So that we will also call $\displaystyle\int_{H^0}$ the natural map $S^*(T_H M) \to S^*(T_{H/H^0} M)$ (this diagram also exists for $\bar{S}^0$ on non-compact manifolds).

The equivariant KK-cycle

In this section we define the transverse cycle associated to a transversally Rockland symbol. It is a class in the equivariant KK-group:

$$\mathrm{KK}_0^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0 M})).$$

This class gives, after some natural operations in KK-theory, a K-homology class in $K^0(C^*(\mathrm{Hol}(H^0)))$. The latter corresponds, in the case where the foliation is a fibration, to the class of some quotient operator on the base of the fibration. The K-homology class thus generally corresponds, in the philosophy of noncommutative geometry, to an abstract Rockland operator on the "space of leaves". Let us first describe the transverse cycle.

Let (E, h) be a hermitian, $\mathrm{Hol}(H^0)$-equivariant, $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over M and let

$$\sigma \in \bar{S}^0(T_{H/H^0} M; E),$$

an order zero odd transverse symbol acting on E which is $\mathrm{Hol}(H^0)$-invariant modulo compact operators and satisfies $\sigma^2 - 1 \in C^*_M(T_{H/H^0} M) \otimes_M \mathrm{End}(E)$ (see the remark below for a justification of this last assumption). The space of compactly supported smooth sections $\Gamma_c(T_{H/H^0} M, \pi^* E \otimes \Omega^{1/2})$ is endowed with a $\Gamma_c(T_{H/H^0} M, \Omega^{1/2})$ inner product:

$$\forall f, g \in \Gamma_c(T_{H/H^0} M, \pi^* E \otimes \Omega^{1/2}), < f, g > (x, \xi) := \int h(f(x, \xi\eta^{-1}), g(x, \eta)),$$

with $\eta \mapsto h(f(x, \xi\eta^{-1}), g(x, \eta))$ defining a density on $T_{H/H^0, x} M$. This product is compatible with the right action by convolution from $\Gamma_c(T_{H/H^0} M, \Omega^{1/2})$:

$$\begin{aligned} \forall f \in \Gamma_c(T_{H/H^0} M, &\pi^* E \otimes \Omega^{1/2}), \forall \gamma \in \Gamma_c(T_{H/H^0} M, \Omega^{1/2}), \\ & f \cdot \gamma(x, \xi) := \int f(x, \xi\eta^{-1}) \gamma(x, \eta) \end{aligned}$$

We can then complete $\Gamma_c(T_{H/H^0}M, \Omega^{1/2})$ into $C^*(T_{H/H^0}M)$. The right module $\Gamma_c(T_{H/H^0}M, \pi^*E \otimes \Omega^{1/2})$ becomes a right Hilbert pre-C^*-module over $C^*(T_{H/H^0}M)$, we denote by $\mathscr{E}$ or $C^*(T_{H/H^0}M, \pi^*E)$ the C^*-module obtained by completion.
Let us denote by $\rho\colon \mathscr{C}_0(M) \to \mathcal{B}_{C^*(T_{H/H^0}M)}(\mathscr{E})$ the morphism induced on smooth sections by

$$\forall a \in \mathscr{C}_c^\infty(M), \forall f \in \Gamma_c(T_{H/H^0}M, \pi^*E \otimes \Omega^{1/2}), \rho(a)(f)(x,\xi) = a(x)\rho(x,\xi).$$

This action makes $(\mathscr{E}, \rho)$ a $(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$-$C^*$-bimodule,indeed ρ is continuous of norm 1 so it extends from $\mathscr{C}_c^\infty(M)$ to $\mathscr{C}_0(M)$. Let us now define $F = \overline{\mathrm{Op}(\sigma)}$.

Theorem 4.2.2. *Let $\sigma \in \bar{S}^0(T_{H/H^0}M; E)$ be an order zero odd transverse symbol acting on E which is $\mathrm{Hol}(H^0)$-invariant modulo compact operators and satisfies $\sigma^2 - 1 \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E)$. The triplet $(\mathscr{E}, \rho, F)$ defines a $\mathrm{Hol}(H^0)$–equivariant Kasparov $(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$-bimodule.*

Proof. By construction of $\bar{S}^0(T_{H/H^0}M)$, we know that

$$F \in \mathcal{M}(C^*(T_{H/H^0}M)) \otimes_M \mathrm{End}(E) = \mathcal{B}_{C^*(T_{H/H^0}M)}(\mathscr{E}).$$

Moreover, the condition $\sigma^2 - 1 \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E)$ implies that $\rho(a)(F^2 - 1)$ is a compact operator for every $a \in \mathscr{C}_0(M)$.

Let $a \in \mathscr{C}_c^\infty(M)$. For a given $x \in M$ $\mathrm{Op}(\sigma_x)$ is a linear operator in the fiber and, restricted to any fiber $\rho(a)$ acts on the as the scalar multiplication by $a(x)$. We thus get that $[\rho(a), F] = 0$. This result extends by continuity to any $a \in \mathscr{C}_0(M)$, i.e. $\forall a \in \mathscr{C}_0(M), [\rho(a), F] = 0$ (in particular these operators are compact).

Finally since σ is $\mathrm{Hol}(H^0)$ equivariant modulo compact operators then F is $\mathrm{Hol}(H^0)$-equivariant modulo compact operators, this means that

$$\forall a \in r^*\mathscr{C}_0(M),\ \rho(a)(\alpha(s^*(F)) - r^*(F)) \in r^*(C^*(T_{H/H^0}M) \otimes_M \mathrm{End}(E)).$$

The triplet $(\mathscr{E}, \rho, F)$ thus defines a $(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$-bimodule and is $\mathrm{Hol}(H^0)$–equivariant in the sense of Le Gall [57]. □

Remark 4.2.3. Let E_0, E_1 be hermitian bundles on which $\mathrm{Hol}(H^0)$ acts by isometries. Let $\sigma_0 \in \bar{S}^0(T_{H/H^0}M, E_0, E_1)$ be a transversally Rockland symbol of order zero, $\mathrm{Hol}(H^0)$-equivariant modulo compact operators. As before we complete $\Gamma_c(E_i)$ into Hilbert-$C^*(T_{H/H^0}M)$ modules (ungraded this time) $\mathscr{E}_i$ $(i = 0, 1)$. Let $E = E_0 \oplus E_1$ be the orthogonal sum with the natural $\mathbb{Z}/2\mathbb{Z}$-grading, the completion of its module of section is $\mathscr{E} = \mathscr{E}_0 \oplus \mathscr{E}_1$. The action of $\mathscr{C}_0(M)$ is the same as before. Finally let $\sigma_1 \in S^0(T_{H/H^0}M, E_1, E_0)$ be a parametrix of σ_0 i.e.

$$\sigma_0 * \sigma_1 - 1 \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E_1)$$

and

$$\sigma_1 * \sigma_0 - 1 \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E_0).$$

Note that σ_1 is also $\mathrm{Hol}(H^0)$-equivariant modulo compact operators.

Let $\sigma = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_0 & 0 \end{pmatrix} \in \bar{S}^0(T_{H/H^0}M, E)$, it is a $\mathrm{Hol}(H^0)$-equivariant (modulo compact operators) transverse symbol with

$$\sigma^2 - 1 \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E),$$

so we are back to our previous setting.

Corollary 4.2.3.1. *If $(\sigma_t)_{t\in[0;1]}$ is a homotopy of symbols in $\bar{S}^0(T_{H/H^0}M)$ satisfying*

$$\forall t \in [0;1], \sigma_t^2 - 1 \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E),$$

then the corresponding path $F_t = \overline{\mathrm{Op}(\sigma_t)}$ gives an operator homotopy between $(\mathscr{E}, \rho, F_0)$ and $(\mathscr{E}, \rho, F_1)$.

Proof. The operator Op is continuous, hence a continuous path of symbols gives a norm continuous path of operators. The additional assumption on the symbols implies that for all $t \in [0;1]$ we have

$$(\mathscr{E}, \rho, F_t) \in \mathbb{E}^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M)).$$

Therefore $(F_t)_{t\in[0;1]}$ is an operator homotopy between the cycles $(\mathscr{E}, \rho, F_0)$ and $(\mathscr{E}, \rho, F_1)$. □

Corollary 4.2.3.2. *The* KK*-theory class*

$$[(\mathscr{E}, \rho, F)] \in \mathrm{KK}_0^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M)),$$

only depends on the class of σ in $\Sigma^0(T_{H/H^0}M; E)$.

Proof. If $f \in C^*_M(T_{H/H^0}M) \otimes_M \mathrm{End}(E)$ then $\sigma_t := \sigma + tf$ satisfies the conditions of the previous corollary, giving an operator homotopy between $(\mathscr{E}, \rho, \mathrm{Op}(\sigma))$ and $(\mathscr{E}, \rho, \mathrm{Op}(\sigma + f))$. □

Definition 4.2.4. We denote by $[\sigma] \in \mathrm{KK}^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$ the KK-theory class described above, it is called the equivariant transverse cycle associated to $\sigma \in \Sigma^0(T_{H/H^0}M; E)$.

Descent and index

Let

$$j_{\mathrm{Hol}(H^0)}\colon \mathrm{KK}_0^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M)) \to \mathrm{KK}_0(C^*(\mathrm{Hol}(H^0)), C^*(\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M))$$

be the descent homomorphism. As described in Section 4.1.2, there is a deformation groupoid $\mathbb{T}_H^{\mathrm{Hol}}M$ from $\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M$ to $M \times M$. In the subsequent parts, we might need to use the fact that

$$\mathrm{ev}_0\colon C^*(\mathbb{T}_H^{\mathrm{Hol}}M_{|[0;1]}) \to C^*(\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M),$$

induces an invertible element in KK-theory. This follows from the contractibility of $C^*(M \times M \times (0;1])$ but we also need the existence of an exact sequence in KK-theory associated with the exact sequence:

$$0 \longrightarrow C^*(M \times M \times (0;1]) \longrightarrow C^*(\mathbb{T}_H^{\mathrm{Hol}}M_{|[0;1]}) \longrightarrow C^*(\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M) \longrightarrow 0.$$

For this we need amenability-type assumption on $\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M$ (the weaker assumption would be K-amenability but in practice, amenability would be easier to verify). Since $T_{H/H^0}M$ is amenable the assumption of (K)-amenability on $\mathrm{Hol}(H^0)$ implies it on $\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M$ so we make it from now on. This result on exact sequences goes back to [49], see also [31]. Such assumptions on holonomy groupoids can be found in [77]. Under such assumptions we get a class

$$\mathrm{Ind}_H^{\mathrm{Hol}} = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \in \mathrm{KK}(\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M, \mathbb{C}),$$

where $\mathrm{ev}_1\colon C^*(\mathbb{T}_H^{\mathrm{Hol}}M_{|[0;1]}) \to C^*(M \times M)$. We then obtain a map:

$$j_{\mathrm{Hol}(H^0)}(\cdot) \otimes \mathrm{Ind}_H^{\mathrm{Hol}}\colon \mathrm{KK}_0^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M)) \to \mathrm{KK}_0(C^*(\mathrm{Hol}(H^0)), \mathbb{C}).$$

We can apply this map to the class of a transversally equivariant transversally Rockland symbol to obtain a K-homology class. This class should correspond to some pseudodifferential operator. In the next sections we construct a K-homology class from a H-pseudodifferential operator whose transverse symbol is $\mathrm{Hol}(H^0)$-equivariant and transversally Rockland. We also show the equality between the resulting K-homology class and the one obtained from the transverse symbol of the operator.

Symbols of positive order

Operators arising in a geometric context are often of positive order. We thus want to be able to compute a K-theory class from symbols of positive order as well. In order to do this, we use the Baaj-Julg picture of KK-theory [6] using unbounded cycles. The Baaj-Julg picture requires regular operators, we need the third part of theorem 2.2.28 and thus restrict to the compact case. In addition to the previous assumptions on the symbols, we further need to assume our symbols to be self-adjoint as it is required in the Baaj-Julg setting. We thus want to prove the following :

Theorem 4.2.5. *Let M be a compact foliated filtered manifold, $E \to M$ be a ${}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$-graded hermitian vector bundle which is $\mathrm{Hol}(H^0)$-equivariant.*
Let $\sigma \in S^m(T_{H/H^0}M; E)$ be an odd, self-adjoint, Rockland symbol of positive order and $\mathrm{Hol}(H^0)$-equivariant modulo smoothing operators. We denote by $D = \overline{\mathrm{Op}}(\sigma)$ the corresponding regular operator on $C^*(T_{H/H^0}M; \pi^*E)$. Then:

$$(C^*(T_{H/H^0}M; \pi^*E), D),$$

is an unbounded Kasparov $(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$-bimodule and its class in $KK^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M))$ does not depend on the self-adjoint representative of the class $\bar{\sigma} \in \Sigma^m(T_{H/H^0}M; E)$.

Proof. The fact that $(C^*(T_{H/H^0}M; \pi^*E), D)$ is an unbounded Kasparov bimodule is rather straightforward to prove. The fact that D is self-adjoint follows from the assumption that $\sigma^* = \sigma$ and as for the bounded case the commutators with elements of $\mathscr{C}_0(M)$ vanish and in particular they are compact. The only thing to show is that $(1+D^2)^{-1} \in \mathcal{K}(C^*(T_{H/H^0}M; \pi^*E))$. Let $q \in S^{-2m}(T_{H/H^0}M; E)$ be a parametrix for σ. Let us denote by

$$Q := \overline{\mathrm{Op}}(q) \in \mathcal{K}(C^*(T_{H/H^0}M; \pi^*E)),$$

the associated operator. Let

$$\rho = \sigma^2 q - 1 \in \Gamma_c(T_{H/H^0}M; \pi^* \mathrm{End}(E) \otimes \Omega^{1/2}),$$

and $R = \overline{\mathrm{Op}}(\rho) \in \mathcal{K}(C^*(T_{H/H^0}M; \pi^*E))$ the associated operator. Since $\mathrm{ord}(\sigma^2) + \mathrm{ord}(q) = 0 \leq 0$ and $\mathrm{ord}(q) = -2m \leq 0$ it follows that

$$\overline{\mathrm{Op}}(\sigma^2 q) = \overline{\mathrm{Op}}(\sigma^2)\overline{\mathrm{Op}}(q) = D^2 Q.$$

From that we get that $1 = R - D^2Q = R - (1 + D^2)Q + Q$ hence

$$(1 + D^2)^{-1} = (1 + D^2)^{-1}R - Q + (1 + D^2)^{-1}Q.$$

Since D is regular we know that $(1 + D^2)^{-1}$ is bounded and the previous equation shows its compactness.

Thus far we have obtained an unbounded Kasparov bimodule. We know from the results of Baaj and Julg that its bounded transform gives a bounded Kasparov module. We thus need to show that the bounded transform of D is equivariant modulo compact operators. Denote by $D_r = \overline{\mathrm{Op}}(r^*\sigma) = r^*D$ and $D_s = s^*D$ the respective pullback operators on $\mathrm{Hol}(H^0)$. We know that $k := r^*\sigma - \alpha(s^*\sigma) \in \Gamma_c(r^*T_{H/H^0}M, \pi^*r^*\operatorname{End}(E)\otimes\Omega^{1/2})$. Let us denote by

$$K \in r^*\mathcal{K}(C^*(T_{H/H^0}M;\pi^*E)),$$

the associated operator. Since k is bounded we have $D_r - \alpha(D_s) = K$. Let $f\colon T \mapsto T(1+T^*T)^{-1/2}$ denote the bounded transform. We need to show that $f(D_r+K) - f(D_r) \in \mathcal{K}_{\mathrm{Hol}(H^0)}(r^*C^*(T_{H/H^0}M;\pi^*E))$. We will use the following facts :

- if Q, S are invertible unbounded operators $Q^{-1} - S^{-1} = Q^{-1}(S-Q)S^{-1}$.
- if Q is a regular operator then

  $$(1+Q^*Q)^{-1/2} = \frac{1}{\pi}\int_0^{+\infty}\lambda^{-1/2}(1+\lambda+Q^*Q)^{-1}\,\mathrm{d}\lambda,$$

  the integral being absolutely convergent.

$$\begin{aligned}
\Delta &:= f(D_r+K) - f(D_r)\\
&= (D_r+K)\left[(1+(D_r+K)^2)^{-1/2} - (1+D_r^2)^{-1/2}\right] + K(1+D_r^2)^{-1/2}\\
&= \frac{1}{\pi}\int_0^{+\infty}\lambda^{-1/2}(D_r+K)\left[(1+\lambda+(D_r+K)^2)^{-1} - (1+\lambda+D_r^2)^{-1}\right]\mathrm{d}\lambda\\
&\quad + K(1+D_r^2)^{-1/2}\\
&= \frac{1}{\pi}\int_0^{+\infty}\lambda^{-1/2}(D_r+K)(1+\lambda+(D_r+K)^2)^{-1}K'(1+\lambda+D_r^2)^{-1}\,\mathrm{d}\lambda\\
&\quad + K(1+D_r^2)^{-1/2}
\end{aligned}$$

where $K' = D_r^2 - (D_r+K)^2 = -D_rK - KD_r - K^2 = -\overline{\mathrm{Op}}(\sigma_r k + k\sigma_r + k^2)$. Since k is smoothing then the second summand of the RHS is compact and we only have to prove that the integral part is compact. First, k being smoothing then so is $\sigma_r k + k\sigma_r + k^2$ and thus

$$K' \in r^*\mathcal{K}(C^*(T_{H/H^0}M;\pi^*E)) \subset \mathcal{K}_{\mathrm{Hol}(H^0)}(r^*C^*(T_{H/H^0}M;\pi^*E)).$$

We then have:

$$(D_r+K)(1+\lambda+(D_r+K)^2)^{-1} = f(D_r+K)(D_r+K)^2)^{-1/2} \in \mathcal{K}_{\mathrm{Hol}(H^0)}(r^*C^*(T_{H/H^0}M;\pi^*E)).$$

Finally the same goes for $(1+\lambda+D_r^2)^{-1}$. We need to check that the integral converges in order to prove that it defines an element of the algebra $\mathcal{K}_{\mathrm{Hol}(H^0)}(r^*C^*(T_{H/H^0}M;\pi^*E))$. We prove it is absolutely convergent. Using functional calculus we prove that:

$$\|(D_r+K)(1+\lambda+(D_r+K)^2)^{-1}\| = \sup_{\mu\in\mathrm{Sp}(D_r+K)} \frac{|\mu|}{1+\lambda+\mu^2} \leq \frac{1}{2\sqrt{1+\lambda}}$$

likewise $\|(1+\lambda+D_r^2)\| \leq \dfrac{1}{1+\lambda}$. Denote by $\lambda \mapsto T(\lambda)$ the integrand. We have

$$\forall \lambda > 0, \|T(\lambda)\| \leq \frac{\|K'\|}{2\sqrt{\lambda}(1+\lambda)^{3/2}}$$

thus the integral converges absolutely and $\Delta \in \mathcal{K}_{\mathrm{Hol}(H^0)}(r^*C^*(T_{H/H^0}M;\pi^*E))$ thus ending the proof. □

4.2.2 The pseudodifferential construction

In this section we give a more explicit construction of the cycle representing $j_{\mathrm{Hol}(H^0)}([\sigma])\otimes \mathrm{Ind}_H^{\mathrm{Hol}}$ with H-pseudodifferential operators. We first recall their definition and then follow the reasoning of [44] appendix A to show that a H-pseudodifferential operator with symbol σ (which, as in the previous section, is transversally Rockland) defines a KK-cycle whose class is:

$$j_{\mathrm{Hol}(H^0)}([\sigma]) \otimes \mathrm{Ind}_H^{\mathrm{Hol}} \in \mathrm{KK}_0(C^*(\mathrm{Hol}(H^0)),\mathbb{C}).$$

The K-homology cycle construction

We now want to construct a K-homology cycle for $C^*(\mathrm{Hol}(H^0))$ from an order 0 H-pseudodifferential operator which is transversally Rockland. The idea is, locally, to induce an operator on some transversal to the foliation. We will get compactness on this transversal using a parametrix. The compactness on the leaves directions will be given by elements of $C^*(\mathrm{Hol}(H^0))$, giving a K-homology cycle. Let us first assume that the foliation is trivial (we will reduce to this case by taking foliated charts) i.e. $M = U\times T$ where the foliation is given by TU and T is a filtered manifold with filtration H'. In this setting we have

$$\begin{aligned}
\mathrm{Hol}(H^0) &= M\times_T M = U\times U\times T\\
\mathbb{T}_H M &= \mathbb{T}U \times_{\mathbb{R}} \mathbb{T}_{H'}T\\
\mathbb{T}_H^{\mathrm{Hol}}M &= (U\times U\times\mathbb{R})\times_{\mathbb{R}}\mathbb{T}_{H'}T = (U\times U)\times\mathbb{T}_{H'}T = \mathrm{Hol}(H^0)\times_T\mathbb{T}_{H'}T.
\end{aligned}$$

The canonical morphism $\varphi\colon \mathbb{T}_H M \to \mathbb{T}_H^{\mathrm{Hol}} M$ is then expressed as the product of the identity of $\mathbb{T}_{H'}T$ and the map $\mathrm{DNC}(\mathrm{Id}_{U\times U})\colon \mathbb{T}U \to U\times U\times\mathbb{R}$. The map $\mathrm{DNC}(\mathrm{Id}_{U\times U})$ comes from the identifications $\mathbb{T}U = \mathrm{DNC}(U\times U, \Delta_U)$ and $U\times U\times\mathbb{R} = \mathrm{DNC}(U\times U, U\times U)$ (see [63] for functoriality of DNC constructions).

For $\lambda > 0$ we write $\alpha_\lambda, \alpha_\lambda^{\mathrm{Hol}}, \alpha_\lambda^{(U)}, \alpha_\lambda^{(T)}$ for the respective actions of $\mathbb{R}_+^*$ on $\mathbb{T}_H M$, $\mathbb{T}_H^{\mathrm{Hol}} M$, $\mathbb{T}U$ and $\mathbb{T}_{H'}T$. We have the relations $\alpha_\lambda = \alpha_\lambda^{(U)} \otimes_{\mathbb{R}} \alpha_\lambda^{(T)}$ and $\alpha_\lambda^{\mathrm{Hol}} = 1\otimes\alpha_\lambda^{(T)}$.
Let $E\to M$ be a hermitian $\mathrm{Hol}(H^0)$-equivariant vector bundle. The holonomy invariance implies that $E = \mathrm{pr}_T^*(E_2)$ for some hermitian vector bundle $E_2 \to T$. Let $\mathscr{E} = C^*(\mathbb{T}_H^{\mathrm{Hol}} M, r^*E)$ and $\mathscr{E}_2 = C^*(\mathbb{T}_{H'}T, r^*E_2)$ we have

$$\mathscr{E} = C^*(U\times U\times\mathbb{R}) \otimes_{\mathscr{C}_0(\mathbb{R})} \mathscr{E}_2 = C^*(\mathrm{Hol}(H^0)) \otimes_{\mathscr{C}_0(T)} \mathscr{E}_2 = C^*(U\times U)\otimes\mathscr{E}_2$$

From a distribution $\mathbb{P} \in \Psi_{H,p}^m(M,E)$ we will induce another distribution $\mathbb{P}' \in \Psi_{H',c}^m(T,E_2)$. In the case $m=0$ we want P to be, in the terminology of [22], a $C^*(\mathrm{Hol}(H^0))$-connection for P' (the non-blackboard letters denote the restriction of the respective distributions on $M\times M$ at time 1).
Let $\eta,\eta' \in \mathscr{C}_c^\infty(\mathrm{Hol}(H^0), \Omega^{1/2})$ and denote by $T_\eta\colon \mathscr{E}_2 \to \mathscr{E}$ the (interior) tensor product by η.

Theorem 4.2.6. *Let $\mathbb{P}' = T_{\eta'}^*\varphi_*(\mathbb{P})T_\eta$. Then we have $\mathbb{P}' \in \Psi_{H',p}^m(T,E_2)$ and $P' := \mathbb{P}'_1 \in \Psi_{H',c}^m(M,E_2)$.*

Let us first explain what $\mathbb{P}'$ does. Let $f,g \in \Gamma_c(\mathbb{T}_{H'}T, r^*E_2\otimes\Omega^{1/2})$ then

$$\begin{aligned}
\langle \mathbb{P}'f, g\rangle &= \langle \varphi_*(\mathbb{P})T_\eta f, T_{\eta'} g\rangle \\
&= \langle \varphi_*(\mathbb{P})(\eta\otimes f), (\eta'\otimes g)\rangle \\
&= \langle \varphi_*(\mathbb{P}), (\eta'\otimes g) * (\eta\otimes f)^*\rangle \\
&= \langle \varphi_*(\mathbb{P}), (\eta' * \eta^*)\otimes(g*f^*)\rangle \\
&= \langle (\eta*\eta'^*\otimes 1)\varphi_*(\mathbb{P}), 1\otimes(g*f^*)\rangle
\end{aligned}$$

Since $\mathrm{pr}_T^* f = f\circ\mathrm{pr}_T = 1\otimes f$ we get

$$\langle \mathbb{P}'f, g\rangle = \langle \mathrm{pr}_{T*}\left((\eta*\eta'^*\otimes 1)\varphi_*(\mathbb{P})\right) f, g\rangle$$

hence

$$\mathbb{P}' = \mathrm{pr}_{T*}(((\eta*\eta'^*)\otimes 1) * \varphi_*(\mathbb{P})).$$

Lemma 4.2.7. *If $\Theta_{\eta,\eta'} = T_{\eta'}^*\cdot T_\eta$ then*

$$\forall\lambda>0, \alpha_{\lambda*}^{(T)}\circ\Theta_{\eta,\eta'} = \Theta_{\eta,\eta'}\circ\alpha_{\lambda*}^{\mathrm{Hol}}$$

Proof. Let $\lambda > 0$, let $\mathbb{Q} \in \mathscr{E}'(\mathbb{T}_H^{\mathrm{Hol}} M, \Omega^{1/2} \otimes r^*E)$. The previous computation shows that

$$\Theta_{\eta,\eta'}(\mathbb{Q}) = \mathrm{pr}_{T*}(((\eta * \eta'^*) \otimes 1) * \mathbb{Q})$$

Using that pr_T commutes with the respective $\mathbb{R}_+^*$-actions we obtain

$$\begin{aligned}\alpha_{\lambda*}^{(T)} \circ \Theta_{\eta,\eta'}(\mathbb{Q}) &= (\alpha_\lambda^{(T)} \circ \mathrm{pr}_T)_*(((\eta * \eta'^*) \otimes 1) * \mathbb{Q}) \\ &= \mathrm{pr}_{T*}(\alpha_{\lambda*}^{\mathrm{Hol}}((\eta * \eta'^*) \otimes 1) * \alpha_{\lambda*}^{\mathrm{Hol}}(\mathbb{Q})) \\ &= \mathrm{pr}_{T*}(((\eta * \eta'^*) \otimes 1) * \alpha_{\lambda*}(\mathbb{Q})) \\ &= \Theta_{\eta,\eta'}(\alpha_{\lambda*}^{\mathrm{Hol}}(\mathbb{Q}))\end{aligned}$$

with $\alpha_{\lambda*}^{\mathrm{Hol}}((\eta * \eta'^*) \otimes 1) = (\eta * \eta'^*) \otimes 1$ because $\alpha_\lambda^{\mathrm{Hol}} \otimes \alpha_\lambda^{(T)}$ and 1 is invariant under the zoom action. ☐

Remark 4.2.8. This lemma shows the usefulness of $\mathbb{T}_H^{\mathrm{Hol}} M$, one could have taken $\eta, \eta' \in \mathscr{C}_c^\infty(\mathbb{T}U, \Omega^{1/2})$, define $T_\eta = \eta \otimes_{\mathscr{C}_0(\mathbb{R})} -$ and induce directly $\mathbb{P}' = T_{\eta'}^* \mathbb{P} T_\eta$. However the same calculations would have yielded

$$\alpha_{\lambda*}^{(T)} \circ \Theta_{\eta,\eta'} = \Theta_{\eta \circ \alpha_{\lambda^{-1}}^{(U)}, \eta' \circ \alpha_{\lambda^{-1}}^{(U)}} \circ \alpha_{\lambda*}.$$

This would then impact the next lemmas. Also using $\mathbb{T}_H^{\mathrm{Hol}} M$ makes the holonomy groupoid appear explicitly in local foliated charts.

Lemma 4.2.9. *$\varphi\colon \mathbb{T}_H M \to \mathbb{T}_H^{\mathrm{Hol}} M$ maps proper subsets to proper subsets.*

Proof. Let $X \subset T_H M$ be a proper subset, $K \subset M \times \mathbb{R}$ a compact set, since φ is a groupoid homomorphism we have:

$$r_{|\varphi(X)}^{-1}(K) = r^{-1}(K) \cap \varphi(X) = \varphi(X \cap r^{-1}(K)) = \varphi(r_{|X}^{-1}(K)).$$

Since X is proper $r_{|X}^{-1}(K)$ is compact and so is $r_{|\varphi(X)}^{-1}(K)$. The same goes for s hence $\varphi(X)$ is proper. ☐

Lemma 4.2.10. *For all $\eta, \eta' \in \mathscr{C}_c^\infty(\mathrm{Hol}(H^0), \Omega^{1/2})$ we have*

$$\Theta_{\eta,\eta'} \circ \varphi_*(\mathscr{C}_p^\infty(\mathbb{T}_H M, \Omega^{1/2}) \otimes r^* \mathrm{End}(E)) \subset \mathscr{C}_p^\infty(\mathbb{T}_{H'} T, \Omega^{1/2}) \otimes r^* \mathrm{End}(E_2).$$

Proof. Let $f \in \Gamma_p(\mathbb{T}_H M, \Omega^{1/2} \otimes r^* \mathrm{End}(E))$, recall that

$$\Theta_{\eta,\eta'}(\varphi_*(f)) = \mathrm{pr}_{T*}(((\eta * \eta') \otimes 1)\varphi_*(f)),$$

hence $\mathrm{supp}(\Theta_{\eta,\eta'}(\varphi_*(f))) = \mathrm{pr}_T(\mathrm{supp}(\eta * \eta'^* \otimes 1) \cap \mathrm{supp}(\varphi_* f))$. We have $\mathrm{supp}(\varphi_*(f)) \subset \varphi(\mathrm{supp}(f))$. Lemma 4.2.9 implies that $\varphi(\mathrm{supp}(f))$ is proper.

Since $\operatorname{supp}(\varphi_*(f))$ is a closed subset contained in a proper subset, it is proper as well. Finally η and η'^* being compactly supported let $K \subset U$ be a compact subset such that $\operatorname{supp}(\eta * \eta'^* \otimes 1) \subset K \times K \times \mathbb{T}_{H'}T$. Let $C \subset T \times \mathbb{R}$ be a compact set

$$\begin{aligned}
r^{-1}_{|\operatorname{supp}(\Theta_{\eta,\eta'}(\varphi_*(f)))}(C) &\subset \operatorname{pr}_T(\operatorname{pr}_T^{-1}(r^{-1}(C)) \cap (K^2 \times \mathbb{T}_{H'}T) \cap \operatorname{supp}(\varphi_*(f))) \\
&\subset \operatorname{pr}_T(K \times K \times r^{-1}(C) \cap \operatorname{supp}(\varphi_*(f))) \\
&\subset \operatorname{pr}_T(r^{-1}(K \times C) \cap \operatorname{supp}(\varphi_*(f))) \\
&\subset \operatorname{pr}_T(r^{-1}_{\operatorname{supp}(\varphi_*(f))}(K \times C)).
\end{aligned}$$

The set $K \times C \subset M \times \mathbb{R}$ is compact and $\operatorname{supp}(\varphi_*(f))$ is proper. Therefore $r^{-1}_{\operatorname{supp}(\varphi_*(f))}(K' \times C)$ is compact and so is $r^{-1}_{|\operatorname{supp}(\Theta_{\eta,\eta'}(\varphi_*(f)))}(C)$ because it is closed. The same reasoning also works for the source map and the set $\operatorname{supp}(\Theta_{\eta,\eta'}(\varphi_*(f)))$ is proper.

We then need to know that the distribution $\Theta_{\eta,\eta'}(\varphi_*(f))$ is a smooth function. In order to do this, we show that its wave front set is empty. Material on wave front sets can be found in [41, 48]. The main result on wave front sets that we will use is Equation (3.6) p.332 of [41]:

$$WF(\varphi_*(f)) \subset \mathrm{d}\varphi_*(WF(f)) \cup N_\varphi,$$

where $N_\varphi = \{[(\varphi(x), \ell)] \in T^*(\mathbb{T}_H^{\mathrm{Hol}}M) \ / \ x \in \mathbb{T}_H M, \ell \circ \mathrm{d}_x\varphi = 0\}$. Since $WF(f) = \emptyset$ we get $WF(\varphi_*(f)) \subset \left[(H^0)^*\right] = PT^*U$ (the wave front set is located at time 0 on the $U \times U$ part of $\mathbb{T}_H^{\mathrm{Hol}}M$) then $\eta * \eta' \otimes 1 \in \mathscr{C}^\infty(\mathbb{T}_H^{\mathrm{Hol}}M, \Omega^{1/2})$ so multiplying by it does not yield a bigger wave front set. Finally since pr_T is a submersion we have $N_{\operatorname{pr}_T} = \emptyset$ and so $WF(\Theta_{\eta,\eta'}(\varphi_*(f))) \subset \operatorname{pr}_T(PT^*U) = \emptyset$ and $\Theta_{\eta,\eta'}(\varphi_*(f)) \in \Gamma(\mathbb{T}_{H'}T, \Omega^{1/2} \otimes r^* \operatorname{End}(E_2))$. □

Proof of the theorem. Since $\varphi \circ r_{\mathbb{T}_H M} = r_{\mathbb{T}_H^{\mathrm{Hol}}M}$, $\varphi_*(\mathbb{P})$ is transverse to $r_{\mathbb{T}_H^{\mathrm{Hol}}M}$, in the same fashion, since pr_T intertwines the range maps of $\mathbb{T}_H^{\mathrm{Hol}}M$ and $\mathbb{T}_{H'}T$ then $\mathbb{P}'$ is transverse to $r_{\mathbb{T}_{H'}T}$.
For quasi-homogeneity, since φ is equivariant with respect to the $\mathbb{R}_+^*$-actions on $\mathbb{T}_H M$ and $\mathbb{T}_H^{\mathrm{Hol}}M$ then for $\lambda > 0$:

$$\begin{aligned}
\alpha_{\lambda*}^{(T)}\mathbb{P}' - \lambda^m\mathbb{P}' &= \alpha_{\lambda*}^{(T)}\Theta_{\eta,\eta'}(\varphi_*(\mathbb{P})) - \Theta_{\eta,\eta'}(\varphi_*(\lambda^m\mathbb{P})) \\
&= \Theta_{\eta,\eta'}\left(\alpha_{\lambda*}^{\mathrm{Hol}}(\varphi_*(\mathbb{P})) - \varphi_*(\lambda^m\mathbb{P})\right) \\
&= \Theta_{\eta,\eta'}\left(\varphi_*(\alpha_\lambda(\mathbb{P}) - \lambda^m\mathbb{P})\right).
\end{aligned}$$

Since $\mathbb{P} \in \Psi_H^m(M)$, we have $\alpha_{\lambda*}(\mathbb{P}) - \lambda^m\mathbb{P} \in \Gamma_p(\mathbb{T}_H M, \Omega^{1/2} \otimes r^* \operatorname{End}(E))$ hence by the last lemma $\alpha_{\lambda*}^{(T)}(\mathbb{P}') - \lambda^m\mathbb{P}' \in \Gamma_p(\mathbb{T}_{H'}T, \Omega^{1/2} \otimes r^* \operatorname{End}(E_2))$, i.e.

$\mathbb{P}' \in \Psi^m_{H'}(T)$. Finally we need to show that $P' = \mathbb{P}'_1$ is compactly supported. Computations similar to the previous ones give that

$$\mathrm{supp}(\mathbb{P}') \subset \mathrm{pr}_T(\mathrm{supp}(\eta * \eta'^* \otimes 1) \cap \varphi(\mathrm{supp}(\mathbb{P}')).$$

Since η and η' are compactly supported then so is $\eta * \eta'^*$. For a fixed $t \neq 0$, $\mathbb{P}'_t = \mathrm{pr}_{T*}((\eta * \eta'^* \otimes 1)\mathbb{P}_t)$ hence $\mathrm{supp}(\mathbb{P}'_t) \cap \Delta_T \subset \mathrm{pr}_T(\mathrm{supp}(\eta * \eta'^* \otimes 1) \cap \Delta_M)$ is compact. Since $\mathbb{P}'$ is properly supported this implies that $\mathrm{supp}(\mathbb{P}'_t) \subset M \times M$ is compact. In particular P' is compactly supported, hence the claim holds true. □

Corollary 4.2.10.1. *If $P \in \Psi^m_H(M; E)$ has symbol $\sigma \in \bar{S}^m(T_H M; E)$ then we have $T^*_{\eta'} P T_\eta \in \Psi^m_{H',c}(T; E_2)$ and its symbol is:*

$$(t, \xi) \mapsto \int_{u \in U} \eta * \eta'^*(u, u, t) \int_{H^0} \sigma(u, t, \xi).$$

Proof. Take any $\mathbb{P} \in \Psi^m_H(M)$ such that $\mathbb{P}_1 = P$ then

$$\mathbb{P}' := T^*_{\eta'} \varphi_*(\mathbb{P}) T_\eta \in \Psi^m_{H'}(T; E_2).$$

However since $\varphi_1 = \mathrm{Id}_{M \times M}$ we have $\mathbb{P}'_1 = T^*_{\eta'} P T_\eta \in \Psi^m_{H'}(T; E_2)$. Let us now compute the symbol of P'. Take $\mathbb{P}$ and $\mathbb{P}'$ as in the theorem then $\mathbb{P}_0 = \sigma$ and $\mathbb{P}'_0 = T^*_{\eta'} \varphi_*(\sigma) T_\eta$. First we compute $\varphi_*(\sigma)$: let

$$f \in \Gamma_c(\mathrm{Hol}(H^0) \ltimes T_{H/H^0} M, \Omega^{1/2} \otimes r^* E_2),$$

recall that here we use the identification

$$\mathrm{Hol}(H^0) \ltimes T_{H/H^0} M = \mathrm{Hol}(H^0) \times_T T_{H'} T,$$

$$\begin{aligned}
\langle \varphi_* \sigma, f \rangle &= \langle \sigma, f \circ \varphi \rangle \\
&= \int_{(u,\delta,t,\xi) \in TU \times T_{H'} T} \sigma(u, \delta, t, \xi) f(u, u, t, \xi) \\
&= \int_{\Delta_U \times T_{H'} T} \left(\int_{H^0} \sigma \right) (u, t, \xi) f(u, u, t, \xi) \\
&= \int_{U \times U \times T_{H'} T} \left(\int_{H^0} \sigma \right) \delta_{\Delta_U} f \\
&= \langle \int_{H^0} \sigma \delta_{\Delta_U}, f \rangle.
\end{aligned}$$

Therefore $\varphi_*(\sigma) = \delta_{\Delta_U} \int_{H^0} \sigma$. We can now compute $\mathbb{P}'_0$. Let

$$g \in \mathscr{C}^\infty(T_{H'} T, \Omega^{1/2} \otimes r^* E_2),$$

$$
\begin{aligned}
\langle \mathbb{P}_0', g\rangle &= \langle \mathrm{pr}_{T*}\left((\eta * \eta'^* \otimes 1)\int_{H^0} \sigma\delta_{\Delta_U}\right), g\rangle \\
&= \langle \int_{H^0} \sigma\delta_{\Delta_U}, \eta' * \eta^* \otimes g\rangle \\
&= \int_{u\in U, t\in T, \xi\in T_{H',x}T} \eta * \eta'^*(u,u,t)\langle \int_{H^0} \sigma(u,t,\xi), g(t,\xi)\rangle
\end{aligned}
$$

hence $\mathbb{P}_0'(t,\xi) = \int_{u\in U} \eta * \eta'(u,u,t) \int_{H^0} \sigma(u,t,\xi)$. □

The next proposition uses the language of connections on Hilbert modules, for background on connections (and the proofs of the results used here) see the appendix of [22].

Proposition 4.2.11. *Let* $\sigma \in \bar{S}^0(T_H M; E)$ *such that* $\int_{H^0} \sigma = 0$. *Let* $P \in \Psi_H^*(M; E)$ *be an operator with symbol* σ. *We have:*

$$\forall F \in C^*(\mathrm{Hol}(H^0)), P(F \otimes_{\mathscr{C}_0(T)} 1), (F \otimes_{\mathscr{C}_0(T)} 1)P \in \mathcal{K}(\mathscr{E}),$$

i.e. P *is a* 0*-connection for* $C^*(\mathrm{Hol}(H^0))$.

Let $\sigma \in \bar{S}^0(T_H M; E)$. *Let us assume that the class modulo* $C^*_M(T_{H/H^0}M) \otimes_M r^* \mathrm{End}(E)$ *of* $\int_{H^0} \sigma$ *contains an element of the form* $1 \otimes \tilde{\sigma}$, *with* $\tilde{\sigma} \in \bar{S}^0(T_{H'}T; E_2)$. *Let* $P \in \Psi_H^*(M; E)$ *and* $\tilde{P} \in \Psi_{H'}^*(T; E_2)$ *of respective symbols* σ *and* $\tilde{\sigma}$. *Then* P *is a* $\tilde{P}$*-connection for* $C^*(\mathrm{Hol}(H^0))$.

Proof. If $\int_{H^0} \sigma = 0$ then, with the same notations as before, P' has a vanishing symbol of order 0. We thus have $P' \in \Psi_{H',c}^{-1}(T; E_2)$. Being of negative order and compactly supported, P' extends to a compact operator on $\mathscr{E}_{2|t=1}$. This works for every $\eta, \eta' \in \mathscr{C}_c^\infty(\mathrm{Hol}(H^0), \Omega^{1/2})$ thus P is a 0-connection. If $F \in C^*(\mathrm{Hol}(H^0))$, then $(F \otimes 1)P, P(F \otimes 1) \in \mathcal{K}(\mathscr{E})$.

Let us assume that $\sigma = 1 \otimes \tilde{\sigma}$. Then the previous corollary gives:

$$\mathbb{P}_0' = \langle \eta, \eta'\rangle\tilde{\sigma}.$$

Therefore we have $P' - \langle \eta, \eta'\rangle\tilde{P} \in \mathcal{K}(\mathcal{E}_{2|t=1})$ (both are compactly supported and they have the same symbol). Then by [22] Remark A.6.4, the operator P is a $\tilde{P}$-connection. □

Let us now return to the general case where the filtration is not necessarily trivial.

Theorem 4.2.12. *Let $E \to M$ be an $\mathrm{Hol}(H^0)$-equivariant, $\mathbb{Z}/2\mathbb{Z}$-graded hermitian bundle. Let $\sigma \in \Sigma^0(T_{H/H^0}M, E)$ be a $\mathrm{Hol}(H^0)$-invariant odd symbol with $\sigma^2 - 1 = 0$. Let $P \in \Psi_H^*(M, E)$ be an operator with transverse symbol σ. Then P induces a K-homology cycle:*

$$(\mathscr{E}, P) \in \mathbb{E}(C^*(\mathrm{Hol}(H^0)), \mathcal{K}(L^2(M))) \cong \mathbb{E}(C^*(\mathrm{Hol}(H^0)), \mathbb{C}).$$

This theorem follows immediately from the following lemma :

Lemma 4.2.13. *If $P \in \Psi_H^*(M, E)$ has vanishing transverse symbol then:*

$$\forall F \in C^*(\mathrm{Hol}(H^0)), P(F \otimes 1), (F \otimes 1)P \in \mathcal{K}(\mathscr{E}).$$

If $P \in \Psi_H^*(M, E)$ has a transverse symbol which is $\mathrm{Hol}(H^0)$-invariant then:

$$\forall F \in C^*(\mathrm{Hol}(H^0)), [P, F \otimes 1] \in \mathcal{K}(\mathscr{E}).$$

Proof. Using partitions of unity, we can see that $C^*(\mathrm{Hol}(H^0))$ is generated by its restriction to foliated charts. We can thus restrict our study to elements $F \in C^*(\mathrm{Hol}(H^0))$ having (compact) support in $\Omega \times_T \Omega$ where $U \times T = \Omega \subset M$ is a foliated chart. We want to use Proposition 4.2.11 but P does not necessarily have support included in Ω^2. Let $\varphi \in \mathscr{C}_c^\infty(\Omega)$ be a cutoff function such that $\varphi F = F\varphi = F$ then:

$$\begin{aligned}
P &= P\varphi + P(1-\varphi) \\
&= P\varphi + \varphi P(1-\varphi) + (1-\varphi)P(1-\varphi) \\
&= P\varphi(2-\varphi) - P\varphi(1-\varphi) + \varphi P(1-\varphi) + (1-\varphi)P(1-\varphi) \\
&= P\varphi(2-\varphi) + [\varphi, P](1-\varphi) + (1-\varphi)P(1-\varphi) \\
&= (P' + K) + K' + (1-\varphi)P(1-\varphi)
\end{aligned}$$

with $P' \in \Psi_{H,c}^0(\Omega, E)$ having symbol $\varphi(2-\varphi)\sigma_H^0(P)$.
The operator $K = P\varphi(2-\varphi) - P'$ is compactly supported and has vanishing order zero symbol, it is then of order -1 hence compact, $K \in \mathcal{K}(\mathscr{E})$. We have $K' = [\varphi, P](1-\varphi)$ and since multiplication by φ is an order 0 operator then $K' \in \Psi_H^0(M, E)$. However,

$$\sigma_H^0(K') = [\varphi, \sigma_H^0(P)](1-\varphi) = 0,$$

since φ is a multiplication by a scalar on each fiber of $T_H M$. Therefore we have $K' \in \Psi_{H,c}^{-1}(M, E)$ and it is compact: $K' \in \mathcal{K}(\mathscr{E})$. Finally since $(1-\varphi)P(1-\varphi)$ and F have disjoint support we get:

$$F(1-\varphi)P(1-\varphi) = (1-\varphi)P(1-\varphi)F = 0.$$

The first part of the lemma hence follows from Proposition 4.2.11 applied to P'. For the second part, consider $\sigma_H^0(P)$ as a symbol on Ω:

$$\sigma_H^0(P)_{|\Omega} \in \bar{S}^0(T_H\Omega, E).$$

This symbol has a transverse part which is $\mathrm{Hol}(H^0)$-invariant and is thus, modulo $C_M^*(T_{H/H^0}\Omega, E)$, of the form $1 \otimes \tilde{\sigma}$, with $\tilde{\sigma} \in \bar{S}^0(T_{H'}T, E')$. According to Proposition 4.2.11, let $P" \in \Psi_H^*(\Omega, E)$ with symbol $\sigma_H^0(P)_{|\Omega}$, then $P"$ is a $\tilde{P}$-connection for $C^*(\mathrm{Hol}(H^0))_{|\Omega}$. In particular $[P", F \otimes 1]$ is compact. Since F commutes with φ then $[\varphi(2-\varphi)P", F\otimes 1] = \varphi(2-\varphi)[\varphi(2-\varphi)P", F\otimes 1]$ is also compact. Finally P' and $\varphi(2-\varphi)P"$ have the same symbol and are compactly supported hence their difference is compact. We thus get that $[P', F\otimes 1]$ is compact and so is $[P, F\otimes 1]$. □

4.2.3 The link between the two constructions

We now want to show that the two KK-classes constructed thus far are equal, we do this using the tangent groupoids. Let E be as before and denote by $\mathscr{E} := C^*(\mathbb{T}_H^{\mathrm{Hol}} M, r^*E)$.
Note that $\mathscr{E}_{|0}$ corresponds to the $C^*(\mathrm{Hol}(H^0) \ltimes T_H M)$–Hilbert module used in the construction of $j_{\mathrm{Hol}(H^0)}([\sigma])$ (with $[\sigma]$ as in Theorem 4.2.2). On the other hand, $\mathscr{E}_{|1}$ corresponds to the $C^*(M\times M)$-Hilbert module used in the construction of the KK-cycle in Theorem 4.2.12. The goal is hence to use some $\mathbb{P} \in \psi_H^0(M)$ and $\mathscr{E}$ to construct a KK-cycle $(\mathscr{E}, \mathbb{P})$ linking the two previous cycles through the evaluation maps ev_0 and ev_1.

Lemma 4.2.14. *If $\mathbb{P} \in \psi_{H,c}^0(M, E)$, then $\mathbb{P}$ extends to an element of $\mathcal{B}_{C^*(\mathbb{T}_H M_{|[0;1]})}(\mathscr{E}_{|[0;1]})$ and $\varphi_*\mathbb{P}$ to an element of $\mathcal{B}_{C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]})}(\mathscr{E}_{|[0;1]})$*

Proof. As before, using a partition of unity, we can reduce to the case where $M = U \times T$, we then have $\mathbb{T}_H^{\mathrm{Hol}} M \cong \mathrm{Hol}(H^0) \times_T \mathbb{T}_{H'} T$. We then need to show that $\mathbb{P}$, being of order 0, extends to a bounded operator on $C^*(\mathbb{T}_H M_{|[-1;1]}, r^*E)$. The proof is standard as it does not differ from the usual pseudodifferential calculus : it follows from the estimates in [81] Corollary 45 which can be taken uniformly for $t \in [-1;1]$. The proof then reduces to the fact that order 0 H-pseudodifferential operators extend to continuous operators on L^2-spaces (which uses the same estimates).
The foliated chart gives maps:

$$\mathrm{pr}_{T*} \colon \mathcal{B}_{C^*(\mathbb{T}_H M)}(C^*(\mathbb{T}_H M, r^*E)) \to \mathcal{B}_{C^*(\mathbb{T}_{H'} T)}(C^*(\mathbb{T}_{H'} T, r^*E_2)),$$

coming from the submersion $\mathrm{pr}_T \colon \mathbb{T}_H M \to \mathbb{T}_{H'} T$ and

$$j \colon \mathcal{B}_{C^*(\mathbb{T}_{H'} T)}(C^*(\mathbb{T}_{H'} T, r^*E_2)) \to \mathcal{B}_{C^*(\mathbb{T}_H^{\mathrm{Hol}} M)}(\mathscr{E}),$$

first defined on compact operators (coming from $\mathbb{T}_H^{\mathrm{Hol}} M = \mathrm{Hol}(H^0) \ltimes \mathbb{T}_{H'} T$) and then extended to the multipliers algebra, i.e. the algebra of bounded operators. As all these morphisms restrict over $[0;1]$, we get the relation $\varphi_* \mathbb{P} = j \circ \mathrm{pr}_{T*} \mathbb{P}$ hence $\varphi_* \mathbb{P} \in \mathcal{B}_{C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]})}(\mathscr{E}_{|[0;1]})$ □

We can thus extend $\Psi_H^0(M,E)_{|[0;1]}$ to a C^*-algebra $\Psi_{H,0}^0(M,E)$ by taking its closure inside $\mathcal{B}_{C^*(\mathbb{T}_H M_{|[0;1]})}(\mathscr{E}_{|[0;1]})$. We now have that:

$$\mathrm{ev}_0(\Psi_{H,0}^0(M,E)) = \bar{S}_0^0(T_H M, E),\ \mathrm{ev}_1(\Psi_{H,0}^0(M,E)) = \Psi_{H,0}^0(M,E).$$

As before we extend this algebra to include operators bounded at infinity, namely:

$$\Psi_H^*(M,E) = \{\mathbb{P} \in \mathcal{B}_{C^*(\mathbb{T}_H M_{|[0;1]})}(\mathscr{E}_{|[0;1]}) \ /\ \forall f \in \mathscr{C}_0([0;1] \times M), f\mathbb{P}, \mathbb{P}f \in \Psi_{H,0}^0(M,E)\}$$

Theorem 4.2.15. *Let $\sigma \in \Sigma^0(T_{H/H^0} M, E)$ be a bounded order 0 transverse symbol,* $\mathrm{Hol}(H^0)$*-invariant with $\sigma^2 = 1$. Let $\mathbb{P} \in \Psi_H^*(M,E)$ with $\int_{H^0} \mathbb{P}_0$ representing σ. Then*

$$(\mathscr{E}_{|[0;1]}, \varphi_* \mathbb{P}) \in \mathbb{E}(C^*(\mathrm{Hol}(H^0)), C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]}))$$

Moreover we have

$$\mathrm{ev}_0([(\mathscr{E}_{|[0;1]}, \varphi_* \mathbb{P})]) = j_{\mathrm{Hol}(H^0)}([\sigma]) \text{ and } \mathrm{ev}_1([(\mathscr{E}_{|[0;1]}, \varphi_* \mathbb{P})]) = [(\mathscr{E}_{|1}, \mathbb{P}_1)].$$

In particular if the foliation is amenable then:

$$[(\mathscr{E}_{|1}, \mathbb{P}_1)] = \mathrm{Ind}_H^{\mathrm{Hol}} \otimes j_{\mathrm{Hol}(H^0)}([\sigma]).$$

Proof of the Theorem. Seeing $\mathscr{E}_{|[0;1]}$ as fibered over $[0;1]$, the algebra $\mathcal{B}_{C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]})}(C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]}, r^*E))$ is identified with the algebra of continuous sections of a C^*-bundle over the compact base $[0;1]$, with fiber at t equal to $\mathcal{B}_{C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|t})}(C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|t}, r^*E))$.
The algebra of compact operators then corresponds to the sections of the sub-C^*-bundle with fibers at t equal to $\mathcal{K}_{C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|t})}(C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|t}, r^*E))$. Because we already know that we are dealing with bounded operators, i.e. continuous sections, the compacity conditions of Kasparov modules can be checked on each fiber. For $t = 0$ this is the result of Theorem 2.2.28 and for $t \in (0;1]$ of Theorem 4.2.12.
We then have:

$$[(\mathscr{E}_{|[0;1]}, \varphi_* \mathbb{P})] \otimes [ev_0] = j_{\mathrm{Hol}(H^0)}(\sigma),$$
$$[(\mathscr{E}_{|[0;1]}, \mathbb{P})] \otimes [ev_1] = [(\mathscr{E}_{|1}, P)],$$

hence the result since $\mathrm{Ind}_H^{\mathrm{Hol}} = [ev_0]^{-1} \otimes [ev_1]$. □

Remark 4.2.16. For the pseudodifferential construction, we did not need any amenability-type assumption on the foliation, nor did we in Theorem 4.2.15. Indeed, take an extension $\mathbb{P} \in \Psi_H^0(M, E)$ of a transversally Rockland operator P with symbol σ. In full generality, we have the class

$$[\sigma] \in \mathrm{KK}^{\mathrm{Hol}(H^0)}(\mathscr{C}_0(M), C^*(T_{H/H^0}M)),$$

from Theorem 4.2.2, the class

$$[P] \in \mathrm{KK}(C^*(\mathrm{Hol}(H^0)), \mathbb{C}),$$

from Theorem 4.2.12 and $[\mathbb{P}] \in \mathrm{KK}(C^*(\mathrm{Hol}(H^0)), C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]}))$. We also have the relations:

$$[\mathbb{P}] \otimes [\mathrm{ev}_0] = j_{\mathrm{Hol}(H^0)}([\sigma])$$
$$[\mathbb{P}] \otimes [\mathrm{ev}_1] = [P]$$

Corollary 4.2.16.1. *In the case of a trivial filtration $H^1 = TM$,*

$$j_{\mathrm{Hol}(H^0)}([\sigma]) \otimes \mathrm{Ind}_H^{\mathrm{Hol}},$$

is the class $\Psi(E, \sigma)$ constructed in [44] Theorem A.7.5.

Proof. In the case of a trivial filtration, the cycle $(\mathscr{E}_{|1}, P)$ is exactly the one of [44]. □

Let us now consider the case where the foliation is a fibration. Let us denote by B its base space and $\pi\colon M \to B$ the projection. The filtration $H^1 \subset \cdots \subset H^r = TM$ induces a filtration $H'^1 \subset \cdots \subset H'^r = TB$ by quotient (and through the identification ${TM}/{H^0} \cong \pi^* TB$). The map p is a Carnot map and we thus have a canonical isomorphism $T_{H/H^0}M \cong \pi^*(T_{H'}B)$. Through this identification, a symbol σ of order k which is holonomy invariant defines a symbol $\pi_*(\sigma) \in S^k(T_{H'}B, E_2)$ which is Rockland if and only if σ is transversally Rockland. Let us then consider a transversally Rockland symbol $\sigma \in \Sigma^0(T_{H/H^0}M, E)$ of order 0 then $\pi_*(\sigma) \in \Sigma^0(T_{H'}B, E_2)$ is a Rockland symbol and hence defines a KK-class $[\pi_*(\sigma)] \in \mathrm{KK}(\mathscr{C}_0(B), C^*(T_{H'}B)$.

Theorem 4.2.17. *Under the isomorphism:*

$$\mathrm{KK}(C^*(\mathrm{Hol}(H^0)), C^*(\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M)) \cong \mathrm{KK}(\mathscr{C}_0(B), C^*(T_{H'}B)),$$

we have:

$$j_{\mathrm{Hol}(H^0)}([\sigma]) = [\pi_*(\sigma)].$$

Moreover the identification

$$\mathrm{KK}(C^*(\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M), C^*(\mathbb{T}_H^{\mathrm{Hol}} M_{|[0;1]})) \cong \mathrm{KK}(C^*(T_{H'}B, C^*(\mathbb{T}_{H'}B_{|[0;1]}))$$

sends $\mathrm{Ind}_H^{\mathrm{Hol}}$ *to the class* $\mathrm{Ind}_{H'}$*. Consequently, if* $P \in \Psi_H^*(M, E)$ *has symbol* σ *then* $\pi_*(P) \in \Psi_{H'}^*(B, E_2)$ *has symbol* $\pi_*(\sigma)$ *and under the isomorphism between the* K*-homology groups of* $C^*(\mathrm{Hol}(H^0))$ *and* B *we get* $[P] = [\pi_*(P)]$*.*

Proof. Let us first extend π to a groupoid morphism $\pi\colon \mathbb{T}_H M \to \mathbb{T}_{H'}B$: over a trivializing subset of the fibration we can write M as $F \times B$ then $\mathbb{T}_H M \cong \mathbb{T}F \times_{\mathbb{R}} \mathbb{T}_{H'}B$ and the morphism is the projection on the second factor. Let $\mathbb{P} \in \Psi_H^0(M, E)$ such that $\mathbb{P}_0 = \sigma$ then since π is a surjective submersion, $\pi_*(\mathscr{C}_p^\infty(\mathbb{T}_H M)) \subset \mathscr{C}_p^\infty(\mathbb{T}_{H'}B)$ and because π is equivariant with respect to the $\mathbb{R}_+^*$-actions on $\mathbb{T}_H M$ and $\mathbb{T}_{H'}B$ we obtain that $\pi_*\mathbb{P} \in \Psi_{H'}^0(B, E_2)$. The class $[\pi_*(\sigma)]$ is then represented by $(\mathscr{E}', \pi_*\mathbb{P}_1')$ which is exactly the image of $(\mathscr{E}, \mathbb{P}_1)$ under the Morita equivalence between $C^*(\mathrm{Hol}(H^0))$ and $\mathscr{C}_0(B)$. □

Remark 4.2.18. We have assumed that the filtrations are 'full' in the sense that there exist an integer $r \geq 0$ such that $H^r = TM$. If it is not the case then the filtration will still eventually become stationary. Denote by r the integer from which the filtration becomes stationary. Then the bracket condition on Lie filtrations forces H^r to be an integrable subbundle, i.e. a foliation. The calculus developped here would then correspond to symbols and operators longitudinal to H^r and transverse to H^0 and the deformation groupoids would have to be replaced by adiabatic deformations of $\mathrm{Hol}(H^r)$ (the same constructions work thanks to the functoriality of those in [63]).
More generally, we can extend our setup to an arbitrary Lie groupoid G with a filtration of its algebroid :

$$\mathcal{A}^0G \subset \mathcal{A}^1G \subset \cdots \subset \mathcal{A}^rG = \mathcal{A}G$$

with the bracket condition $\forall i, j \geq 0 \left[\Gamma(\mathcal{A}^iG), \Gamma(\mathcal{A}^jG)\right] \subset \Gamma(\mathcal{A}^{i+j}G)$. The calculus obtained is a G-filtered calculus and the addition of $\mathcal{A}^0G$ would then allow to define a transverse symbols and a transversal Rockland condition for these operators. The reasoning would be the same as what has been done so for the sake of comprehensiveness and not having too heavy notations we restricted ourselves to the case of $G = M \times M$ and a 'full' filtration.

4.2.4 A link with equivariant index for countable discrete group actions

Let M be a filtered manifold with filtration $H^1 \subset \cdots \subset H^r = TM$ and Γ be a countable discrete group acting on M by filtration preserving diffeomorphims.

Let P be a manifold with fundamental group Γ [2]. Let $\tilde{P} \to P$ be the universal cover of P. The manifold $\tilde{X} = \tilde{P} \times M$ is foliated by the fibration $\mathrm{pr}_M \colon \tilde{X} \to M$ (i.e. the leaves are the $\tilde{P} \times \{m\}$ for $m \in M$) let us denote by $\tilde{H}^0 = T\tilde{P} \times M$ the corresponding integrable subbundle. The filtration on M induces a filtration on $\tilde{X}$ with $\tilde{H}^i = T\tilde{P} \times H^i$ for $i \geq 1$ so that $(\tilde{H}^i)_{i\geq 0}$ is a foliated filtration on $\tilde{X}$. Γ acts diagonally on $\tilde{X}$ and preserves the foliation and the filtration. Moreover since the action $\Gamma \curvearrowright \tilde{P}$ is free and proper then so is the diagonal action $\Gamma \curvearrowright \tilde{X}$. Let $X = \tilde{P} \times_\Gamma M$ be the quotient manifold. Since the foliated filtration was Γ-invariant, it induces a foliated filtration $(\bar{H}^i)_{i\geq 0}$ on X. We have $T_{\tilde{H}}\tilde{X} = TX \times T_H M$ thus $T_{\tilde{H}/\tilde{H}^0}\tilde{X} \cong \tilde{P} \times T_H M$. From this we deduce that, for any $m \in M$, the projection $\tilde{X} \to X$ induces an isomorphism $T_{H,m}M \cong T_{\bar{H}/\bar{H}^0,[p,m]}X$. The resulting foliation is made up to replace the group action, more precisely, since the foliation on $\tilde{X}$ is given by the projection on M then $\mathrm{Hol}(\tilde{H}^0)$ is Morita equivalent to M. From this get the Morita equivalence $\mathrm{Hol}(\bar{H}^0) \sim \Gamma \ltimes M$.

Let us now consider $\sigma \in \Sigma(T_H M; E)^\Gamma$ an equivariant symbol on an equivariant hermitian bundle E. The pullback by pr_M induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*_M(T_H M; E) & \longrightarrow & \bar{S}^0(T_H M; E) & \longrightarrow & \Sigma^0(T_H M; E) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \mathrm{pr}_M^*} & & \downarrow {\scriptstyle \mathrm{pr}_M^*} & & \downarrow {\scriptstyle \mathrm{pr}_M^*} & & \\ 0 & \longrightarrow & C^*_{\tilde{X}}(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E) & \longrightarrow & \bar{S}^0(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E) & \longrightarrow & \Sigma^0(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E) & \longrightarrow & 0 \end{array}$$

in the category of Γ-C^*-algebras. Let us define $\tilde{\sigma} \in \Sigma^0(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E)$ as $\mathrm{pr}_M^*(\sigma)$, it is equivariant with respect to the diagonal action. If $q \colon \tilde{X} \to X$ denotes the canonical projection then we get the same type of diagrams as before:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*_X(T_{\bar{H}/\bar{H}^0}X; \bar{E}) & \longrightarrow & \bar{S}^0(T_{\bar{H}/\bar{H}^0}X; \bar{E}) & \longrightarrow & \Sigma^0(T_{\bar{H}/\bar{H}^0}X; \bar{E}) & \longrightarrow & 0 \, . \\ & & \downarrow {\scriptstyle q^*} & & \downarrow {\scriptstyle q^*} & & \downarrow {\scriptstyle q^*} & & \\ 0 & \longrightarrow & C^*_{\tilde{X}}(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E) & \longrightarrow & \bar{S}^0(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E) & \longrightarrow & \Sigma^0(T_{\tilde{H}/\tilde{H}^0}\tilde{X}; \mathrm{pr}_M^* E) & \longrightarrow & 0 \end{array}$$

Moreover, should we replace all the algebras on the bottom line by the fixed points algebras for the Γ action then every vertical arrow would become an isomorphism. The symbol $\tilde{\Sigma}$ thus factors into a symbol $\bar{\sigma} \in \Sigma^0(T_{\bar{H}/\bar{H}^0}X, \bar{E})$ where $\bar{E}$ corresponds to the quotient bundle of $\mathrm{pr}_M^* E$ on X. Because of the previous isomorphims for the fibers of $T_{\bar{H}/\bar{H}^0}X$ and the construction of $\bar{\sigma}$, we get that $\bar{\sigma}$ is transversally Rockland if and only if σ is Rockland. Let us

[2] This is always possible by taking open subsets in $\mathbb{R}^5$. If the group is finitely presented we can even assume P to be compact.

assume $\sigma^2 = 1$ hence $\bar{\sigma}^2 = 1$. We then are under the conditions of theorem 4.2.2 and get a class

$$[\bar{\sigma}] \in \mathrm{KK}^{\mathrm{Hol}(\bar{H}^0)}(\mathscr{C}_0(X), C^*(T_{\bar{H}/\bar{H}^0}X))$$

One could also directly see that σ itself defines an equivariant class

$$[\sigma] \in \mathrm{KK}^{\Gamma \ltimes M}(\mathscr{C}_0(M), C^*(T_H M))$$

The Morita equivalence $\mathrm{Hol}(\bar{H}^0) \sim \Gamma \ltimes M$ sends $\mathscr{C}_0(X)$ to $\mathscr{C}_0(M)$ and $C^*(T_{\bar{H}/\bar{H}^0}X)$ to $C^*(T_H M)$, giving an isomorphism

$$\mathrm{KK}^{\mathrm{Hol}(\bar{H}^0)}(\mathscr{C}_0(X), C^*(T_{\bar{H}/\bar{H}^0}X)) \cong \mathrm{KK}^{\Gamma \ltimes M}(\mathscr{C}_0(M), C^*(T_H M))$$

sending $[\bar{\sigma}]$ to $[\sigma]$.

Remark 4.2.19. The previous sections can be applied to prove the following : let $P \in \Psi_H^*(M; E)$ be an operator on M with symbol σ, then P is equivariant modulo compact operators and defines a KK-class $[P] \in \mathrm{KK}^{\Gamma}(\mathscr{C}_0(M), \mathbb{C})$. Applying Thom isomorphism twice and Kasparov's second Poincaré duality theorem [50] we get an isomorphism

$$\mathrm{KK}^{\Gamma \ltimes M}(\mathscr{C}_0(M), C^*(T_H M)) \cong \mathrm{KK}^{\Gamma}(\mathscr{C}_0(M), \mathbb{C}).$$

It would be interesting to show how $[\sigma]$ and $[P]$ are intertwined through this isomorphism.

Chapter 5

Transversal BGG sequences

5.1 Kostant's Hodge theory and the origins of the BGG machinery

We first recall the classical construction Kostant's laplacian for a semi-simple Lie group G and a parabolic subgroup P and the associated Hodge theory. This construction was introduced in [54] as a way to prove the Borel-Weil-Bott theorem (a way to construct irreducible representations of a certain highest weight for a semi-simple Lie group) and is purely algebraic. It involves the Lie algebra homology and cohomology introduced in [17, 55] to perform a construction similar to the Hodge theory in Riemannian geometry. This analogy has a more concrete incarnation as the differential will be the local model (i.e. symbol) of a differential operator on the homogeneous space (and later on spaces with parabolic geometries). Let G be a semi-simple Lie group and $P \subset G$ a parabolic subgroup, denote by $\mathfrak{g}$ and $\mathfrak{p}$ their respective Lie algebras. We also assume the Lie algebra $\mathfrak{g}$ to be $|k|-$graded i.e.

$$\mathfrak{g} = \bigoplus_{i=-k}^{k} \mathfrak{g}_i, \ \forall i, j, [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

such that $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$. Such decomposition always exists and only needs an appropriate choice of Cartan subalgebra and positive root system (see [82] for instance). The Killing form on $\mathfrak{g}$ being non-degenerate, it induces isomorphisms of $\mathfrak{g}_0$-modules $\mathfrak{g}_i \cong \mathfrak{g}^*_{-i}$ for $i \neq 0$. In particular we get $\left(\mathfrak{g}/\mathfrak{p}\right)^* \cong \mathfrak{p}_+$ where $\mathfrak{p}_+ = \bigoplus_{i>0} \mathfrak{g}_i$. On the other hand we can identify the quotient space $\mathfrak{g}/\mathfrak{p}$ to the Lie algebra $\mathfrak{g}_- = \bigoplus_{i<0} \mathfrak{g}_i$. These two identifications allow us to use the co-chain complex of $\mathfrak{g}_-$ and the chain complex for $\mathfrak{p}_+$ at the same time. Indeed, let $\mathbb{E}$ be a $\mathfrak{g}$-module so it can be restricted to a $\mathfrak{g}_-$-module and a $\mathfrak{p}_+$ module at the same time. Recall that the Chevalley-Eilenberg complex construction endows $\Lambda^\bullet\mathfrak{g}_-^* \otimes \mathbb{E}$ with the structure of a co-chain complex with a de Rham type differential ∂ as co-boundary map. In a dual fashion, the complex $\Lambda^\bullet\mathfrak{p}_+ \otimes \mathbb{E}$ has a structure of a chain complex with boundary map ∂^*. The duality result invoked earlier gives for each $n \geq 0$ an isomorphism $C_k(\mathfrak{p}_+, \mathbb{E}) := \Lambda^k\mathfrak{p}_+ \otimes \mathbb{E} \cong \Lambda^k\mathfrak{g}_-^* \otimes \mathbb{E} =: C^k(\mathfrak{g}_-, \mathbb{E})$. Under these identification the maps ∂ and ∂^* become adjoint to each other for natural inner products and hermitian forms on $\mathfrak{g}^*$ and E, see [54][1]. The map ∂^* is called the Kostant co-differential. We can then form Kostant's laplacian:

$$\Box_\bullet := \partial^*\partial + \partial\partial^* \colon C_\bullet(\mathfrak{p}_+, \mathbb{E}) \to C_\bullet(\mathfrak{p}_+, \mathbb{E}).$$

Lemma 5.1.1. *Let $k \geq 0$ and $x \in \Lambda^k \left(\mathfrak{g}/\mathfrak{p}\right)^* \otimes \mathbb{E}$. If $\partial\partial^* x = 0$ then $\partial^* x = 0$. If $\partial^*\partial x = 0$ then $\partial x = 0$.*

Proof. Follows directly from the fact that the Killing form is positive definite and ∂ and ∂^* are adjoint maps. □

We can now perform a Hodge-type decomposition of the complex:

Theorem 5.1.2 (Kostant [54])**.** *Let $n \geq 0$, we have the following decomposition, orthogonal for the Killing form:*

$$\Lambda^k \left(\mathfrak{g}/\mathfrak{p}\right)^* \otimes \mathbb{E} = \mathrm{im}(\partial) \oplus \ker(\Box_n) \oplus \mathrm{im}(\partial^*).$$

Every (co)homology class in $H^k(\mathfrak{g}_-, \mathbb{E})$ or $H_k(\mathfrak{p}_+, \mathbb{E})$ has a unique harmonic (i.e. in $\ker(\Box_k)$) representative, this provide $\mathfrak{g}_0$-equivariant sections of the natural projections $C^k(\mathfrak{g}_-, \mathbb{E}) \to H^k(\mathfrak{g}_-, \mathbb{E})$ and $C_k(\mathfrak{p}_+, \mathbb{E}) \to H_k(\mathfrak{p}_+, \mathbb{E})$.

Proof. We view $\Lambda^k \left(\mathfrak{g}/\mathfrak{p}\right)^* \otimes \mathbb{E}$ both as k-chains on $\mathfrak{p}_+$ and k-cochains on $\mathfrak{g}_-$. To show the decomposition we first show that $\ker(\Box_k) = \ker(\partial) \cap \ker(\partial^*)$. If $x \in \ker(\Box_k)$ then we have $\partial\Box_k x = 0$ which gives $\partial\partial^*\partial x = 0$. Applying the previous lemma twice gives $\partial x = 0$. The same reasoning with $\partial^*\Box_k x$ gives $\partial^* x = 0$ thus the inclusion $\ker(\Box_k) \subset \ker(\partial) \cap \ker(\partial^*)$. The converse inclusion is obvious by construction of $\Box_k$. Now another easy consequence of the lemma is that $\mathrm{im}(\partial) \cap \ker(\partial^*) = 0$ and $\mathrm{im}(\partial^*) \cap \ker(\partial) = 0$. Combining these results we get $(\mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*)) \cap \ker(\Box_k) = 0$. We now have the three subspaces in direct sum and need to prove that they generate the whole space. From the inclusion $\mathrm{im}(\Box_k) \subset \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*)$ and the rank nullity

[1] This justifies the notation ∂^* for the homological differential.

theorem applied to $\Box_k$ we get the equality of dimensions between the sum of subspaces and the whole space and thus the Hodge decomposition. It follows from the decomposition that $\ker(\partial^{(*)}) = \ker(\Box_k) \oplus \mathrm{im}(\partial^{(*)})$ and thus the isomorphisms (of $\mathfrak{g}_0$-modules):

$$H_k(\mathfrak{p}_+, \mathbb{E}) \cong \ker(\Box_k) \cong H^k(\mathfrak{g}_-, \mathbb{E}).$$

□

From this result we can explain the BGG machinery. The goal is to create a new complex that will compute the same cohomology. Denote by $\pi_k\colon \ker(\partial) \to H_k(\mathfrak{p}_+, \mathbb{E})$ the natural projection and by $S_k\colon H_k(\mathfrak{p}_+, \mathbb{E}) \xrightarrow{\sim} \ker(\Box_k) \subset C^k(\mathfrak{g}_-, \mathbb{E})$ its section constructed in the previous theorem. The idea is now to define the Bernstein-Gelfand-Gelfand operator as:

$$D_k := \pi_{k+1}\partial_k S_k\colon H_k(\mathfrak{p}_+, \mathbb{E}) \to H_{k+1}(\mathfrak{p}_+, \mathbb{E})$$

However on the algebraic level since $\ker(\Box_k) = \ker(\partial) \cap \ker(\partial^*)$ this just gives $D_k = 0$. Another way to see that this idea is too naive at the level of the Lie algebra is to use Kostant's decomposition of $H_k(\mathfrak{p}_+, \mathbb{E})$ as a direct sum of irreducible representations of highest weight for $\mathfrak{g}_0$. Since $H_k(\mathfrak{p}_+, \mathbb{E})$ and $H_{k+1}(\mathfrak{p}_+, \mathbb{E})$ involve different irreducible representations, there cannot be non-trivial homomorphisms of $\mathfrak{g}_0$-modules between them. The trick at the algebraic level is to use the Verma modules associated to these representations instead. This was the original idea of Bernstein, Gelfand and Gelfand in [10]. On the homogeneous space, Verma modules can be seen as homogeneous bundles using jets bundles. This observation allowed for a geometric formulation of the BGG machinery on manifolds with parabolic geometry in [83]. In this context the Kostant co-differential is still a bundle map but the differential ∂ becomes a differential operator (the de Rham differential with respect to some connection). Combining these two operators will give rise to a Hodge theory on differential forms and allow us to define the BGG operators as previously. This time however they will give a sequence of (non-trivial) differential operators. In the flat case (for the aforementioned connection) the de Rham sequence of operators is an actual co-chain complex and the BGG sequence is a complex as well. In this case, both compute the same cohomology.

5.2 The geometric setting

We fix G a semi-simple Lie group and $P \subset G$ a parabolic subgroup. We also assume the Lie algebra $\mathfrak{g}$ to be $|k|-$graded i.e.

$$\mathfrak{g} = \bigoplus_{i=-k}^{k} \mathfrak{g}_i, \ \forall i, j, [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

such that $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$. Such decomposition always exists and only needs an appropriate choice of Cartan subalgebra and positive root system (see [82] for instance). If M is a manifold endowed with a right P-action, we denote by r the morphism $r\colon P \to \mathrm{Diff}(M)^{op}$. By differentiation this induces a map $\mathfrak{p} \to \mathfrak{X}(M)$. We denote this map by $\xi \mapsto X_\xi$, we have:

$$\forall \xi \in \mathfrak{p}, \forall x \in M, X_\xi(x) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} r_{\exp(t\xi)}(x).$$

Note that this map corresponds to the the anchor of the action algebroid $M \rtimes \mathfrak{p} \to M$.

Definition 5.2.1. A transverse $(G, P)-$geometry is the data of $(M, \mathcal{F}, \omega)$ where M is a manifold, $\mathcal{F} \subset TM$ is an integrable subbundle (i.e. a foliation) and $\omega\colon TM \to \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form with the following properties :

(i) there is a proper smooth right action of P on M such that $\mathcal{F}$ is P-invariant and $M \rtimes \mathfrak{p} \subset \mathcal{F}$ [2]

(ii) $\ker(\omega) \subset \mathcal{F}$

(iii) ω is P-equivariant i.e. $\forall h \in P, r_h^*\omega = Ad(h^{-1}) \circ \omega$

(iv) $\forall \xi \in \mathfrak{p}, \omega(X_\xi) = \xi$

(v) $\overline{\omega}\colon {}^{TM}\!/_{\mathcal{F}} \to M \times {}^{\mathfrak{g}}\!/_{\mathfrak{p}}$ is an isomorphism

(vi) $\forall X \in \Gamma(\ker(\omega)), L_X\omega = 0$

Example 5.2.2. *If $\Gamma \subset G$ is a discrete subgroup then $\Gamma \backslash G$ has a transverse (G, P)-geometry for the Maurer-Cartan form. Here $\mathcal{F}$ is the foliation induced by the P-action. The leaf space then corresponds to $\Gamma \backslash G/P$.*

[2] We mean that the image of $M \rtimes \mathfrak{p}$ by its anchor is included in $\mathcal{F}$. Condition (iv) below implies that the P-action is locally free. We can thus identify $M \rtimes \mathfrak{p}$ to its image by its anchor.

This definition is different from the usual one [11]. In the literature a (foliated) principal P-bundle over M is added to the data, our setting makes the total space of this bundle the relevant object. The relevant object in both settings is the "space of leaves" $M/\mathcal{F}$ (i.e. the geometry of the transversals) and the base foliation or the one on the total space of the bundle have the same transverse structure. The idea is thus, for the bundle of spaces of leaves:

$$M/\ker(\omega) \to M/\mathcal{F},$$

to mimic "locally" the principal P bundle $G \to G/P$ with its Maurer-Cartan connection form.

Moreover, our setting allows non-free actions of P but axiom (iv) makes it locally free giving M/P an orbifold structure. Moreover taking $M \rtimes \mathfrak{p} = \mathcal{F}$ and $M \curvearrowleft P$ free gives a principal P-bundle, its base is endowed with a parabolic geometry (in the sense of Cartan). This is the usual definition of non-foliated parabolic geometries that is used in the literature, see [83, 82, 27] for instance.

Example 5.2.3. *We give the geometric structures on the space of leaves corresponding to the pair* (G, P)*:*

- $G = PGL(n+1)$ *and* P *the isotropy group of a line correspond to projective geometry*
- $G = O(n+1, 1)$ *and* P *the stabilizer of a point on the conformal sphere correspond to conformal geometry*
- $G = SU(p+1, q+1)$ *and* P *the stabilizer of an isotropy line in* $\mathbb{C}^{p+q+2}$ *correspond to CR geometry*
- $G = SP(n+1)$ *and* P *the stabilizer of a line in* $\mathbb{R}^{2(n+1)}$ *correspond to projective contact geometry*

Proposition 5.2.4. $\ker(\omega)$ *is a foliation.*

Proof. Let $X, Y \in \Gamma(\ker(\omega))$,

$$\begin{aligned}
\omega([X,Y]) &= X \cdot \omega(Y) - Y \cdot \omega(X) - \mathrm{d}\omega(X,Y) \\
&= -\mathrm{d}\omega(X,Y) \\
&= -\iota_X \mathrm{d}\omega(Y) \\
&= -L_X \omega(Y) \\
&= 0.
\end{aligned}$$

□

Corollary 5.2.4.1. *The foliation $\mathcal{F}$ splits into the direct sum:*

$$\mathcal{F} = \ker(\omega) \oplus M \rtimes \mathfrak{p}.$$

Both are subalgebroids.

Proof. By axiom (*iv*) we know the sum is direct. We already know the inclusion $\supset$. The other is a consequence of axiom (*v*). $\square$

Definition 5.2.5. The curvature form of ω is $K = d\omega + [\omega, \omega]_{\mathfrak{g}} \in \Omega^2(M, \mathfrak{g})$.

Let us now define the filtration of TM induced by the transverse parabolic geometry and the associated osculating nilpotent Lie algebra bundle. For $i \in \mathbb{Z}$ let $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$ ($\cdots \supset \mathfrak{g}^{i-1} \supset \mathfrak{g}^i \supset \mathfrak{g}^{i+1} \supset \cdots$). We then have the relations $\forall i, j \in \mathbb{Z}, [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ and $\forall i, \mathfrak{g}_i = \mathfrak{g}^i / \mathfrak{g}^{i+1}$. Define a Lie bracket with:

$$\begin{aligned}[\cdot,\cdot]\colon \mathfrak{g}^{-i}/\mathfrak{g}^{-i+1} \times \mathfrak{g}^{-j}/\mathfrak{g}^{-j+1} &\to \mathfrak{g}^{-i-j}/\mathfrak{g}^{-i-j+1}\\ (X,Y) &\mapsto [X,Y]_{\mathfrak{g}} \mod \mathfrak{g}^{-i-j+1}.\end{aligned}$$

It endows $\bigoplus_{i \leq 0} \mathfrak{g}_i$ with a Lie algebra structure (which is not the one induced from $\mathfrak{g}$ since $\mathfrak{p}_+$ is not an ideal in $\mathfrak{g}$). $\mathfrak{g}_0$ is then an ideal and the quotient is isomorphic to $\mathfrak{g}_- = \bigoplus_{i<0} \mathfrak{g}_i$. Note that $\mathfrak{p}$ is not an ideal in $\mathfrak{g}$ hence the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p} = \mathfrak{g}_-$ is not a Lie algebra homomorphism.

Let $TM = H^{-r} \supset \cdots \supset H^{-1} \supset H^0$ be a foliated filtration i.e.

$$\forall i, j \left[\Gamma(H^{-i}), \Gamma(H^{-j})\right] \subset \Gamma(H^{-i-j}).$$

Using the Lie bracket of vector fields we get in the same fashion as before

$$[\cdot,\cdot]\colon \Gamma(H^{-i})/\Gamma(H^{-i+1}) \times \Gamma(H^{-j})/\Gamma(H^{-j+1}) \to \Gamma(H^{-i-j})/\Gamma(H^{-i-j+1})$$

This bracket is tensorial in both variables. With the identifications

$$\Gamma(H^{-i})/\Gamma(H^{-i+1}) = \Gamma\left(H^{-i}/H^{-i+1}\right),$$

for $i \geq 1$, we get a fiberwise Lie bracket $[\cdot,\cdot]\colon \mathfrak{t}_H M \to \mathfrak{t}_H M$. Here we have denoted:

$$\mathfrak{t}_H M = H^{-1} \oplus H^{-2}/H^{-1} \oplus \cdots \oplus H^{-r}/H^{-r+1}.$$

The algebroid $\mathfrak{t}_H M$ is a bundle of nilpotent Lie algebras (*a priori* not locally trivial). Denote by $T_H M$ the bundle of (connected, simply connected) nilpotent Lie groups integrating them using the Baker-Campbell-Hausdorff formula.

For all $x \in M$, $H^0_x \subset \mathfrak{t}_{H,x} M$ is included in the center. We denote by $\mathfrak{t}_{H/H^0} M$ the quotient Lie algebra bundle and $T_{H/H^0} M$ the corresponding Lie group bundle. Back to our setting, the Cartan connection $\overline{\omega}$ yields an identification of vector bundles:

$$TM/\mathcal{F} \cong M \times \mathfrak{g}/\mathfrak{p}.$$

The Lie brackets on vector fields and on $\mathfrak{g}$ might however fail to be compatible. Requiring ω to be a Lie algebra homomorphism would be too strong, however we can define a filtration on TM and get weaker conditions in order to have $\mathfrak{t}_{H/H^0} M \cong M \times \mathfrak{g}_-$. Let $\overline{H}^{-i} = \overline{\omega}^{-1}\left(\mathfrak{g}^{-i}/\mathfrak{p}\right) \subset TM/\mathcal{F}$ and $H^{-i} \subset TM$ its lift, in particular $H^0 = \mathcal{F}$ and $H^{-k} = TM$.

Definition 5.2.6. A (G,P)-transverse geometry is called regular if $(H^{-i})_{i \geq 0}$ is a foliated Lie filtration and $\mathfrak{t}_{H/H^0} M \cong M \times \mathfrak{g}_-$[3] as bundles of Lie algebras.

Lemma 5.2.7.

$$\forall i \geq 0, \left[\Gamma(H^{-i}), \Gamma(H^0)\right] \subset \Gamma(H^{-i})$$

Proof. Let $i \geq 0$ and $X \in \Gamma(H^0), Y \in \Gamma(H^{-i})$ we need to show that $\omega([X,Y]) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i})$. We have

$$\omega([X,Y]) = X \cdot \omega(Y) - Y \cdot \omega(X) - \mathrm{d}\omega(X,Y),$$

but $\omega(Y) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i})$ and $\mathfrak{g}^{-i}$ being a vector subspace of $\mathfrak{g}$, then

$$X \cdot \omega(Y) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i}).$$

Likewise, $\omega(X) \in \mathscr{C}^\infty(M, \mathfrak{p}) \subset \mathscr{C}^\infty(M, \mathfrak{g}^{-i})$ as $-i \leq 0$ so we also have

$$Y \cdot \omega(X) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i}).$$

Finally $\mathrm{d}\omega(X,Y) = K(X,Y) - [\omega(X), \omega(Y)]_{\mathfrak{g}}$. We already know that the action of $\mathfrak{p}$ preserves the filtration: $[\mathfrak{g}^i, \mathfrak{p}] \subset \mathfrak{g}^i$. We thus need to show that $K(X,Y) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i})$. We show a stronger result that will be useful later on. □

Lemma 5.2.8.

$$\forall X \in \Gamma(\mathcal{F}), \iota_X K = 0$$

Proof. We use the fact that $\mathcal{F} = M \rtimes \mathfrak{p} \oplus \ker(\omega)$.

[3] As P-equivariant vector bundles where the action $P \curvearrowright \mathfrak{g}_-$ is trivial.

First case, $X \in \ker(\omega)$ **:** We have $\omega(X) = 0$ so the bracket part vanishes and $\iota_X \, \mathrm{d}\omega = L_X \omega = 0$ by axiom (vi), hence $\iota_X K = 0$.

Second case, $X = X_\xi, \xi \in \mathfrak{p}$ **:** $\iota_{X_\xi}\omega$ is a constant function so $\mathrm{d}\iota_{X_\xi}\omega = 0$ and $\iota_{X_\xi} \, \mathrm{d}\omega = L_{X_\xi}\omega$.

$$\begin{aligned} L_{X_\xi}\omega &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} r^*_{\exp(t\xi)}\omega \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} Ad(\exp(-t\xi)) \circ \omega \\ &= -ad(\xi) \circ \omega \\ &= -[\xi, \omega] \\ &= -[\omega(X_\xi), \omega] \\ &= -\iota_{X_\xi}[\omega, \omega]_{\mathfrak{g}}. \end{aligned}$$

Therefore $\iota_{X_\xi} K = -\iota_{X_\xi}[\omega, \omega]_{\mathfrak{g}} + \iota_{X_\xi}[\omega, \omega]_{\mathfrak{g}} = 0$. □

Proposition 5.2.9. *i)* $(H^{-i})_{i \geq 1}$ *is a Lie filtration if and only if:*

$$\forall i, j \geq 1, K(H^{-i}, H^{-j}) \subset \mathfrak{g}^{-i-j}.$$

ii) The (G, P)*-transverse geometry is regular if and only if:*

$$\forall i, j \geq 1, K(H^{-i}, H^{-j}) \subset \mathfrak{g}^{-i-j+1}.$$

Proof. Let $i, j \geq 1$, $X \in \Gamma(H^{-i}), Y \in \Gamma(H^{-j})$:

$$\begin{aligned} \omega([X, Y]) &= X \cdot \omega(Y) - Y \cdot \omega(X) - \mathrm{d}\omega(X, Y) \\ &= X \cdot \omega(Y) - Y \cdot \omega(X) - [\omega(X), \omega(Y)] + K(X, Y). \end{aligned}$$

We have that $X \cdot \omega(Y) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-j}) \subset \mathscr{C}^\infty(M, \mathfrak{g}^{-i-j})$. We also have that $Y \cdot \omega(X) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i-j})$. Therefore we get $[\omega(X), \omega(Y)] \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i-j})$ and thus deduce the first statement.

For the second statement we keep the same notations. The only thing to show is that $\omega([X, Y]) = [\omega(X), \omega(Y)] \mod \mathfrak{g}^{-i-j+1}$. Since $i, j \geq 1$ we have $\mathscr{C}^\infty(M, \mathfrak{g}^{-j}) \subset \mathscr{C}^\infty(M, \mathfrak{g}^{-i-j+1})$ so $X \cdot \omega(Y) \in \mathscr{C}^\infty(M, \mathfrak{g}^{-i-j+1})$. The same goes for $Y \cdot \omega(X)$ and we have

$$\omega([X, Y]) = [\omega(X), \omega(Y)] + K(X, Y) \mod \mathfrak{g}^{-i-j+1},$$

hence the result. □

5.3 The transverse complex

Let $(\mathbb{E}, \rho)$ be a finite dimensional $\mathfrak{g}$-representation, denote by $E \to M$ the associated trivial bundle and by $\nabla = \mathrm{d} + \rho \circ \omega \in \Omega^1(M, E)$ the canonical tractor connection. We want to mimic the curved BGG sequences of [83, 82]. In order to do so we need the identification of the normal bundle with $\mathfrak{g}/\mathfrak{p}$ and thus replace the usual complex of differential forms by the one of transverse forms. Let $N \subset TM$ be a subbundle such that $\mathcal{F} \oplus N = TM$, such a choice has no incidence on what follows. The space of differential forms then splits up into a bicomplex

$$\Lambda^{i,j} T^* M = \Lambda^i \left({TM}/{F} \right)^* \otimes \Lambda^j \mathcal{F}^* \cong \Lambda^i N^* \otimes \Lambda^j \mathcal{F}^*.$$

The de Rham differential splits up accordingly:

$$\mathrm{d}^\nabla = \sum_{i+j=1} \mathrm{d}^\nabla_{i,j},$$

with $\mathrm{d}^\nabla_{i,j}(\Omega^{k,l}(M,E)) \subset \Omega^{k+i,l+j}(M,E)$. Since d^∇ is determined by its image on zero-forms and one-forms then only the following terms remain:

$$\mathrm{d}^\nabla = \mathrm{d}^\nabla_{2,-1} + \mathrm{d}^\nabla_{1,0} + \mathrm{d}^\nabla_{0,1} + \mathrm{d}^\nabla_{-1,2}.$$

We denote by $\mathrm{d}^\nabla_N = \mathrm{d}^\nabla_{1,0}$ the differential in the transverse directions, $\mathrm{d}^\nabla_{\mathcal{F}} = \mathrm{d}^\nabla_{0,1}$ the differential in the longitudinal ones. For a vector field $X \in \mathfrak{X}(M)$, write $X = X^N + X^F$ for the decomposition corresponding to $\mathfrak{X}(M) = \Gamma(N) \oplus \Gamma(\mathcal{F})$.

Proposition 5.3.1. $\mathrm{d}^\nabla_{-1,2} = 0$ *and* $\mathrm{d}^\nabla_{2,-1}$ *is the contraction by* $(2,-1)$*-form* $\Theta \in \Gamma(M, \Lambda^2 N^* \otimes \mathcal{F})$ *given by* $\Theta(X^N, Y^N) = -[X^N, Y^N]^F$*. In particular* $\mathrm{d}^\nabla = \mathrm{d}^\nabla_N + \mathrm{d}^\nabla_{\mathcal{F}}$ *if and only if* N *is involutive.*

Proof. As mentioned before, we only need to study the case of zero and one-forms. Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M. Let $s \in \Gamma(E)$,

$$\mathrm{d}^\nabla s(X) = \mathrm{d}^\nabla s(X^N) + \mathrm{d}^\nabla s(X^F) = \mathrm{d}^\nabla_N s(X^N) + \mathrm{d}^\nabla_{\mathcal{F}} s(X^F).$$

Let $\omega \in \Omega^{1,0}(M,E)$, we view ω as a 1-form vanishing on $\mathcal{F}$, then

$$\begin{aligned} \mathrm{d}^\nabla \omega(X,Y) &= \mathrm{d}^\nabla \omega(X^N, Y^N) + \mathrm{d}^\nabla \omega(X^N, Y^F) \\ &\quad + \mathrm{d}^\nabla \omega(X^F, Y^N) + \mathrm{d}^\nabla \omega(X^F, Y^F) \\ &= \mathrm{d}^\nabla_N \omega(X^N, Y^N) + \mathrm{d}^\nabla_{\mathcal{F}} \omega(X^N, Y^F) \\ &\quad + \mathrm{d}^\nabla_{\mathcal{F}} \omega(X^F, Y^N) + \mathrm{d}^\nabla_{-1,2} \omega(X^F, Y^F) \end{aligned}$$

hence $\mathrm{d}^{\nabla}_{-1,2}\omega(X^F, Y^F) = \nabla_{X^F}\omega(Y^F) - \nabla_{Y^F}\omega(X^F) - \omega([X^F, Y^F]) = 0$ as $\mathcal{F}$ is involutive.
Since $\mathrm{d}^{\nabla}_{-1,2}$ vanishes automatically on $\Omega^{0,1}(M, E)$ we get the first part of the statement.
For the second part, notice that we necessarily have $\mathrm{d}^{\nabla}_{2,-1}\omega = 0$ on $\Omega^{1,0}(M, E)$, so does the contraction by Θ. Let $\omega \in \Omega^{0,1}(M, \mathbb{E})$ i.e. a 1-form vanishing on N. We have

$$\begin{aligned}
\mathrm{d}^{\nabla}_{2,-1}\omega(X^N, Y^N) &= \mathrm{d}^{\nabla}\omega(X^N, Y^N) \\
&= \nabla_{X^N}\omega(Y^N) - \nabla_{Y^N}\omega(X^N) - \omega([X^N, Y^N]) \\
&= -\omega([X^N, Y^N]^N + [X^N, Y^N]^F) \\
&= \omega(\Theta(X^N, Y^N))
\end{aligned}$$

Hence the result. □

The object of study in the next sections will be the sequence of differential operators:

$$\Gamma(E) \xrightarrow{\mathrm{d}^{\nabla}_N} \Omega^{1,0}(M, E) \xrightarrow{\mathrm{d}^{\nabla}_N} \cdots \xrightarrow{\mathrm{d}^{\nabla}_N} \Omega^{n,0}(M, E)$$

As for the de Rham sequence of operators, this is not a chain complex, $(\mathrm{d}^{\nabla}_N)^2 \neq 0$. The definition of d^{∇} involves choosing a complementary bundle to F but the operator pulled back between the bundles $\Lambda^i \left({TM}/{H^0}\right)^* \otimes E$ does not depend on these choices. Denote by R^{∇} the curvature form associated to ∇, it is the image of K by $\mathfrak{g} \to \mathrm{End}(\mathbb{E})$.

Corollary 5.3.1.1. *We have* $(\mathrm{d}^{\nabla}_{\mathcal{F}})^2 = 0, (\mathrm{d}^{\nabla}_N)^2 = R^{\nabla} - (\mathrm{d}^{\nabla}_F\iota_\Theta + \iota_\Theta\,\mathrm{d}^{\nabla}_F), \iota_\Theta^2 = 0$

Proof. The proof is similar to the previous, we square on both sides

$$\mathrm{d}^{\nabla} = \mathrm{d}^{\nabla}_N + \mathrm{d}^{\nabla}_{\mathcal{F}} + \iota_\Theta,$$

and identify the (i, j)-parts for $-4 \leq i, j \leq 4, i + j = 2$. The LHS is equal to R^{∇}. By 5.2.8 R^{∇} is a form of bidegree $(2, 0)$. Hence we get the following equalities:

(4,-2) : $\iota_\Theta^2 = 0$

(3,-1) : $\mathrm{d}^{\nabla}_N\iota_\Theta + \iota_\Theta\,\mathrm{d}^{\nabla}_N = 0$

(2,0) : $(\mathrm{d}^{\nabla}_N)^2 + \mathrm{d}^{\nabla}_{\mathcal{F}}\iota_\Theta + \iota_\Theta\,\mathrm{d}^{\nabla}_{\mathcal{F}} = R^{\nabla}$

(1,1) : $\mathrm{d}^{\nabla}_{\mathcal{F}}\,\mathrm{d}^{\nabla}_N + \mathrm{d}^{\nabla}_N\,\mathrm{d}^{\nabla}_{\mathcal{F}} = 0$

$$(0,2) : (\mathrm{d}_{\mathcal{F}}^{\nabla})^2 = 0$$

$$(-1,3) : 0 = 0$$

$$(-2,4) : 0 = 0$$

□

Remark 5.3.2. This sequence of operators can also be constructed for arbitrary foliated filtered manifold. The curvature is not concentrated in degree (2,0) in general however. The same type of result that the last corollary still holds with the (2,0)-component of the curvature (and some other terms do not vanish anymore but they are not relevant for our study).

5.4 The holonomy action

The de Rham complex of a manifold with transverse parabolic geometry has no chance of being elliptic, even hypoelliptic, as it fails to capture what happens in the longitudinal directions. We actually have better chances to understand its index properties transversally to $\mathcal{F}$. In order to do so we need actions of the holonomy groupoid and the equivariance of the operators involved. The problem is that the holonomy groupoid only acts on the normal bundle to the foliation. To avoid this issue we will make sections of the algebroid act on the different bundles. For the P part of the holonomy groupoid it can be reduced to a classical P-equivariance.

Definition 5.4.1. Let $D\colon \Gamma(M,E) \to \Gamma(M,F)$ be a differential operator between $\mathrm{Hol}(F)$-equivariant vector bundles E, F over M. We say that D is $\mathrm{Hol}(F)$-equivariant if for every (local) section $X \in \Gamma(M,F)$ then:

$$\exp(X) \circ D = D \circ \exp(X).$$

Proposition 5.4.2. *If* $D\colon \Gamma(M,E) \to \Gamma(M,F)$ *is* $\mathrm{Hol}(F)$*-equivariant then its principal transverse symbol (whether in the classical or filtered calculus if there is a filtration) is* $\mathrm{Hol}(F)$*-equivariant in the usual sense.*

Proof. The holonomy groupoid acts on the transverse bundle. Therefore the equivariance on the symbol can be localized from exponentials of local sections of F to elements of the groupoid. We thus get the usual notion of equivariance. □

Recall from 5.2 that M has a transversally filtered structure induced by its transverse Cartan geometry. We will now assume that the transverse Cartan geometry is regular. It is proven in chapter 4 that the action $\mathrm{Hol}(H^0) \curvearrowright {}^{TM}\!/_{H^0}$ can be refined to an action $\mathrm{Hol}(H^0) \curvearrowright T_{H/H^0}M$ preserving the groupoid structure of $T_{H/H^0}M$. Here we have decomposed the foliation into two parts $H^0 = \mathcal{F} = M \rtimes \mathfrak{p} \oplus \ker(\omega)$. To understand the different actions of $\mathrm{Hol}(H^0)$ in our case, we will thus need to understand the actions of P and $\mathrm{Hol}(\ker(\omega))$.

The holonomy groupoid acts on E in the following manner : P acts on M on the right via r and on $\mathbb{E}$ on the left via ρ. It endows E with the diagonal action : $(x,e)\cdot h = (r_h(x), \rho(h^{-1})(e))$. Similarly $\mathrm{Hol}(\ker(\omega))$ acts naturally on M and trivially on $\mathbb{E}$ and the action on E is the diagonal one. Let P act on $\mathfrak{g}$ on the left by the restriction of the adjoint action. Then, P acts on the trivial bundle $M \times \mathfrak{g}$ by $(x,\xi)\cdot h = (r_h(x), \mathrm{ad}(h^{-1})(\xi))$, it makes $\omega\colon TM \to M \times \mathfrak{g}$ a P-equivariant morphism by axiom (iii). We make $\mathrm{Hol}(\ker(\omega))$ act trivially on $\mathfrak{g}$ and thus obtain an action of $\mathrm{Hol}(H^0)$ on $M \times \mathfrak{g}$ (which extends to the associated bundles $M \times G, M \times \mathfrak{g}^*, \cdots$).

Remark 5.4.3. The action of $\mathrm{Hol}(\ker(\omega))$ on E or $\mathrm{End}(E)$ is given by the pullback of sections : $s \cdot \exp(X) = s \circ \exp(X) = \exp(X)^* s$.

Lemma 5.4.4. *$\overline{\omega}$ is $\mathrm{Hol}(H^0)$-equivariant, hence $\rho\circ\omega$ is $\mathrm{Hol}(H^0)$ equivariant.*

Proof. The equivariance for the P action is clear from the previous paragraph. We now need to prove the equivariance for elements of the form $\exp(X)$ with $X \in \Gamma(\ker(\omega))$.

Let X be a section of $\ker(\omega)$, $t \mapsto \gamma_t = \exp(tX)$ the associated flow. Since $\gamma_{t*}X \in \Gamma(\ker(\omega))$ then $\exp(X)^*\bar{\omega} = \exp(0)^*\bar{\omega} = \bar{\omega}$. The equivariance then follows from the action of $\mathrm{Hol}(\ker(\omega))$ on the fibers of E being trivial. □

Proposition 5.4.5. *∇ is a $\mathrm{Hol}(H^0)$-equivariant connection*

Proof. We consider $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$. The action on the later space of sections is the following : for elements h of P we have $\eta\cdot h = \mathrm{Ad}(\rho(h^{-1}))r_h^*\eta$, for elements $\exp(X)$ with $X \in \Gamma(\ker(\omega))$ we have $\eta \cdot \exp(X) = \exp(X)^*\eta$.

First case, the P-action: Let $h \in P$, we have $s \cdot h = \mathrm{Ad}(\rho(h^{-1}))r_h^* s$. Since the P-action is constant on the fibers of E we have :

$$\begin{aligned}
\mathrm{d}(s\cdot h) &= \mathrm{d}(\mathrm{Ad}(\rho(h^{-1}))r_h^* s) \\
&= \mathrm{Ad}(\rho(h^{-1}))(\mathrm{d}r_h^* s) \\
&= \mathrm{Ad}(\rho(h^{-1}))r_h^*\,\mathrm{d}s \\
&= (\mathrm{d}s)\cdot h.
\end{aligned}$$

The equivariance of ∇ with respect to the P-action is thus a consequence of the previous lemma.

Second case, the $\mathrm{Hol}(\ker(\omega))$**-action** The action of an element of $\mathrm{Hol}(H^0)$ is given by the pullback of sections and thus commutes with d. The equivariance of ∇ is again a consequence of the lemma. $\square$

Corollary 5.4.5.1. *The differential operators*

$$\mathrm{d}_N^\nabla \colon \Omega^{i,0}(M,E) \to \Omega^{i+1,0}(M,E), i \geq 0,$$

are $\mathrm{Hol}(H^0)$*-equivariant.*

5.5 Graded filtered operators and the graded transverse Rockland condition

5.5.1 Graded order in the filtered calculus

In this section we combine the results of chapter 4 with the setting of graded filtered calculus of [27]. A vector space $\mathbb{E}$ is filtered if it admits a (finite) filtration $\cdots \supseteq \mathbb{E}^{j-1} \supseteq \mathbb{E}^j \supseteq \cdots$ into subspaces such that $\mathfrak{g}_i\mathbb{E}^j \subset \mathbb{E}^{i+j}$. This is always the case if $\mathbb{E}$ is a finite dimensional $\mathfrak{g}$-module for a $|k|$-graded Lie algebra $\mathfrak{g}$. Indeed there is then an element $X_0 \in \mathfrak{g}_0$ such that $\mathrm{ad}(X_0)_{|\mathfrak{g}_i} = i\,\mathrm{Id}_{\mathfrak{g}_i}$, diagonalizing the image of X_0 gives the decomposition of $\mathbb{E}$. Moreover in this case the action of $\mathfrak{g}$ preserves the filtration in the sense that $\mathfrak{g}_i\mathbb{E}^j \subset \mathbb{E}^{i+j}$. If a vector space is filtered, we denote by $\mathrm{gr}(\mathbb{E})$ its associated graded vector space, i.e. $\mathrm{gr}(\mathbb{E}) = \bigoplus_p \mathrm{gr}_p(\mathbb{E})$ with $\mathrm{gr}_p(\mathbb{E}) = \mathbb{E}^p/\mathbb{E}^{p+1}$. The graded vector space is naturally filtered and $\mathrm{gr}(\mathrm{gr}(\mathbb{E})) = \mathrm{gr}(\mathbb{E})$. If $\mathbb{E}$ and $\mathbb{F}$ are filtered and $f \colon \mathbb{E} \to \mathbb{F}$ preserves the filtration then there is a naturally defined linear map $\mathrm{gr}(f) \colon \mathrm{gr}(\mathbb{E}) \mapsto \mathrm{gr}(\mathbb{F})$. A splitting of $\mathbb{E}$ is a filtration preserving isomorphism of vector spaces $\mathrm{gr}(\mathbb{E}) \xrightarrow{\sim} \mathbb{E}$ whose associated graded homomorphism is the identity. In the case of a $\mathfrak{g}$ action as earlier, the subalgebra $\mathfrak{g}_-$ acts on $\mathrm{gr}(\mathbb{E})$ and all the morphisms above preserve the $\mathfrak{g}_-$-module structure. All these notions transpose *mutati mutandis* to vector bundles over a topological space.

During this section M denotes a filtered manifold with

$$H^{-1} \subset \cdots \subset H^{-r} = TM$$

and E, F vector bundles over M, graded in the previous sense. A pseudodifferential operator of graded filtered order m is an operator $P \colon \Gamma(E) \to \Gamma(F)$

such that for any splittings S_E, S_F of E and F we have[4]:

$$\forall p, q, (S_F^{-1} P S_E)_{q,p} \in \Psi_H^{m+q-p}(M; \mathrm{gr}_p(E), \mathrm{gr}_q(F)).$$

It is actually enough to show this for a single choice of splittings. We denote by $\tilde{\Psi}_H(M; E, F)$ the space of such operators. We can then define their principal symbol as:

$$\tilde{\sigma}^m(P) = \sum_{p,q} \sigma^{m+q-p}(S_F^{-1} P S_E) \in \bigoplus_{p,q} \Sigma^{m+q-p}(T_H M; \mathrm{gr}_p(E), \mathrm{gr}_q(F)),$$

the later space will be denoted $\widetilde{\Sigma}^m(T_H M; \mathrm{gr}(E), \mathrm{gr}(F))$. This symbol does not depend on the choice of splitting. Since $\mathrm{gr}(E)$ and $\mathrm{gr}(F)$ are graded they are endowed with a family of inhomogeneous dilations in the same way $T_H M$ is. We denote them by δ_λ^E and δ_λ^F respectively, $\lambda > 0$. The principal symbols of graded order m are then kernels k that satisfy the homogeneity condition:

$$\forall \lambda > 0, \delta_{\lambda *} k \circ \delta_\lambda^E = \lambda^m \delta_\lambda^F \circ k.$$

For differential operators, recall that the filtration of TM induces an isomorphism of graded algebras $\mathrm{Diff}_H(M) \cong \mathcal{U}(\mathfrak{t}_H M)$. In regard of this new notion of order, when filtered vector bundles are involved, one should consider $\mathcal{U}(\mathfrak{t}_H M) \otimes \hom(\mathrm{gr}(E), \mathrm{gr}(F))$ instead with the graduation given by the graded tensor product i.e. its k-th stratum is:

$$\bigoplus_{p,q} \mathcal{U}(\mathfrak{t}_H M)_{-k+q-p} \otimes \hom(\mathrm{gr}_p(E), \mathrm{gr}_q(F)).$$

It is shown in [27] that this calculus satisfies all the usual properties of a pseudodifferential calculus (one basically goes back to the same properties for the usual filtered calculus). In particular they construct an appropriate Sobolev scale and get continuity results for operators in this calculus.

An operator $P \in \widetilde{\Psi}_H^m(M; E, F)$ is said to be graded Rockland if for every $x \in M$ and every non-trivial irreducible unitary representation of the fiber $\pi \in \widehat{T_{H,x} M} \setminus \{1\}$, the operator

$$\mathrm{d}\pi(\tilde{\sigma}(P)) \colon \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(E_x) \to \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(F_x)$$

is left injective.

Finally for differential operators, the notion of graded order is simpler. Indeed, let $D \colon \Gamma(M, E) \to \Gamma(M, F)$ be a differential operator of graded filtered order m. Since there are no differential operator of negative order then

[4] For differential operator, replace Ψ_H^m by the differential operators of order m in the filtered calculus.

$(S_F^{-1}DS_E)_{q,p} = 0$ whenever $m + q - p < 0$. Consequently D maps $\Gamma(M, E^p)$ to $\Gamma(M, F^{p-m})$ and we get an associated graded operator

$$\mathrm{gr}_k(D) \colon \Gamma(M, \mathrm{gr}_\bullet(E)) \to \Gamma(M, \mathrm{gr}_{\bullet-m}(F)).$$

Moreover this operator is tensorial and its corresponding vector bundle homomorphism is the direct sum $\bigoplus\limits_p \tilde{\sigma}^m(D)_{p-m,p}$. In particular a differential operator of graded order 0 preserves the filtration and we will denote by $\widetilde{D} \colon \mathrm{gr}\, E \to \mathrm{gr}(F)$ the homomorphism constructed from the symbol. In the foliated case, when writing gr for bundles involving TM, we will consider H^0 as part of H^1 so that the associated graded is a bundle of nilpotent Lie algebras[5], i.e. $\mathrm{gr}(TM) = \mathfrak{t}_H M$.

Now if M has a foliated filtration, all those notions carry out to transversal symbols and we can define the classes of transverse principal symbols $\widetilde{\Sigma}^m(T_{H/H^0}M; \mathrm{gr}(E), \mathrm{gr}(F))$. We also get as in chapter 4 restriction maps:

$$\int_{H^0} \colon \widetilde{\Sigma}^m(T_H M; \mathrm{gr}(E), \mathrm{gr}(F)) \to \widetilde{\Sigma}^m(T_{H/H^0}M; \mathrm{gr}(E), \mathrm{gr}(F))$$

that are compatible with the product of symbols.

Definition 5.5.1. An operator $P \in \tilde{\Psi}_H^m(M; E, F)$ is transversally graded Rockland if for every $x \in M$ and $\pi \in \widehat{T_{H/H^0}M} \setminus \{1\}$

$$\mathrm{d}\pi \left(\int_{H^0} \tilde{\sigma}(P) \right) \colon \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(E_x) \to \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(F_x),$$

is injective.

5.5.2 Graded (transversal) Rockland sequences

The differential operators we want to analyse are not single operators but a sequence of them. We need to replace the Rockland condition on operators by a condition on sequences of operators. This is done in the same way elliptic complexes are defined for the usual pseudodifferential calculus (see [4]). Let us consider a sequence of vector bundles $E_i \to M$ and pseudodifferential operators A_i of order k_i in the filtered calculus:

$$\cdots \xrightarrow{A_{i-2}} \Gamma(M, E_{i-1}) \xrightarrow{A_{i-1}} \Gamma(M, E_i) \xrightarrow{A_i} \Gamma(M, E_{i+1}) \xrightarrow{A_{i+1}} \cdots .$$

[5] See chapter 4, if we consider H^0 in the grading process, the groupoid becomes $\mathrm{Hol}(H^0) \ltimes T_{H/H^0}M$.

This sequence is a Rockland sequence if for every $x \in M$ and every non-trivial irreducible unitary representation $\pi \in \widehat{T_{H,x}M} \setminus \{1\}$ the sequence:

$$\cdots \longrightarrow \mathcal{H}_\pi^\infty \otimes E_{i,x} \xrightarrow{\mathrm{d}\pi(\sigma_x^{k_i}(A_i))} \mathcal{H}_\pi^\infty \otimes E_{i+1,x} \longrightarrow \cdots$$

is weakly exact. This means that the image of an arrow is contained and dense in the kernel of the next one. We can replace the Rockland condition with the graded one, replacing the condition on the $\sigma^{k_i}(A_i)$ by the same condition of weak exactness for the sequence of operators:

$$\cdots \longrightarrow \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(E_{i,x}) \xrightarrow{\mathrm{d}\pi(\tilde{\sigma}_x^{k_i}(A_i))} \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(E_{i+1,x}) \longrightarrow \cdots .$$

Like before we can adapt this definition with the transversal symbols. We now take M to be a foliated filtered manifold.

Definition 5.5.2. A sequence of operators

$$\cdots \xrightarrow{A_{i-2}} \Gamma(M, E_{i-1}) \xrightarrow{A_{i-1}} \Gamma(M, E_i) \xrightarrow{A_i} \Gamma(M, E_{i+1}) \xrightarrow{A_{i+1}} \cdots$$

with $A_i \in \widetilde{\Psi}_H^{k_i}(M, E_i, E_{i+1})$ is transversally graded Rockland if for every $x \in M$ and $\pi \in \widehat{T_{H/H^0,x}M} \setminus \{1\}$ the sequence

$$\cdots \longrightarrow \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(E_{i,x}) \xrightarrow{\mathrm{d}\pi \int_{H^0} \tilde{\sigma}_x^{k_i}(A_i)} \mathcal{H}_\pi^\infty \otimes \mathrm{gr}(E_{i+1,x}) \longrightarrow \cdots$$

is weakly exact.

In particular for a sequence of operators to be (transversal/graded) Rockland, a necessary condition is that the sequence of operators obtained from the (transversal/graded) symbols has to be a complex.

5.6 Graded transverse Rockland property of the transverse complex

5.6.1 The general case

Let G be a semi-simple Lie group, P a parabolic subgroup and (M, F) a manifold with transverse parabolic (G, P)-geometry. We consider $\mathbb{E}$ a G-representation and $E \to M$ the associated bundle. We take ∇ the corresponding tractor connection on E and the sequence of operators

$$(\Omega^{\bullet,0}(M, E), \mathrm{d}_N^\nabla),$$

introduced in section 5.3.

The goal of this section is to prove that the transverse complex is graded transversally Rockland. To do this we will identify the graded symbol of d^{∇} to the differential of the cohomological complex of $\mathfrak{g}_-$ acting on $\mathbb{E}$ and will need the regularity assumption. We will prove more general results for foliated filtered manifolds with graded vector bundle and connections in the spirit of [27]. Those results will hold under some assumptions that will automatically hold for regular transverse parabolic geometries when the vector bundle is a tractor bundle and the connection the associated tractor connection.

Let M be a foliated filtered manifold, $E \to M$ a filtered vector bundle and ∇ a linear connection on E that preserves the filtration. By that we mean that if $X \in \Gamma(H^p), p \leq 0$ and $\xi \in E^q$ then $\nabla_X \xi \in \Gamma(E^{p+q})$[6]. Define $\omega := \mathrm{gr}(\nabla) \in \Gamma(M, \mathfrak{t}^*_H M \otimes \mathrm{gr}(E))$[7]. Since ∇ is filtration preserving, ω takes values in the space of elements of degree 0 (for the graduation of the graded tensor product). Sections of H^0 are considered as degree 1 so their image under ω vanishes. We can thus push it forward to $\overline{\omega} \in \Gamma(M, \mathfrak{t}^*_{H/H^0} M \otimes \mathrm{gr}(E))$. More generally if $A\colon E \to T^*M \otimes E$ preserves the filtration then images of sections of H^0 vanish under $\mathrm{gr}(A)$ so we can factor it to

$$\overline{\mathrm{gr}(A)}\colon\ \mathrm{gr}(E) \to \mathfrak{t}^*_{H/H^0} M \otimes \mathrm{gr}(E)).$$

We can extend the construction of the transverse complex of section 5.3 in this case so we fix a subbundle $N \subset TM$ complementary to H^0. Since ∇ is filtration preserving, the operators d^{∇}_N are differential operators of graded filtered order 0.

We have identifications

$$\mathrm{gr}(\Lambda^k T^*M \otimes E) \cong \Lambda^k \mathfrak{t}^*_H M \otimes \mathrm{gr}(E)$$

and

$$\mathrm{gr}(\Lambda^{k,0} T^*M \otimes E) \cong \Lambda^k \mathfrak{t}^*_{H/H^0} M \otimes \mathrm{gr}(E).$$

Let $x \in M$,we have the identification:

$$\mathscr{C}^\infty(T_{H/H^0,x}M, \mathrm{gr}(\Lambda^{k,0} T^*_x M \otimes E_x)) \cong \Omega^k(T_{H/H^0,x}M, \mathrm{gr}(E_x))$$

because the tangent bundle of the group $T_{H/H^0,x}M$ is trivial.

[6]Recall that here the convention is that the H^p are defined for non-positive p, thus the connection lowers the degree in the fibers of E.

[7]For transverse parabolic geometries it corresponds to $\rho \circ \omega$ where $\rho\colon \mathfrak{g} \to \mathrm{End}(\mathbb{E})$ is the representation inducing the tractor bundle.

Lemma 5.6.1. *Under the identification*

$$\mathscr{C}^\infty(T_{H/H^0,x}M, \mathrm{gr}(\Lambda^{k,0}T_x^*M \otimes E_x)) \cong \Omega^k(T_{H/H^0,x}M, \mathrm{gr}(E_x))$$

we have:

$$\tilde{\sigma}_x^{0,\perp}(\mathrm{d}_N^\nabla) = \mathrm{d} + \overline{\omega} \wedge_x \cdot.$$

Proof. It is immediate from the explicit formula for d^∇ that $\tilde{\sigma}_x^0(\mathrm{d}^\nabla)$ corresponds as a map $\Omega^k(T_{H,x}M, \mathrm{gr}(E_x)) \to \Omega^{k+1}(T_{H,x}M, \mathrm{gr}(E_x))$. Restricting to the normal directions thus gives the result for the transverse symbol. □

Since d_N^∇ is a differential operator of order 0 we can also consider $\mathrm{gr}(\mathrm{d}_N^\nabla)$ and taking its quotient $\partial^{\overline{\omega}} := \overline{\mathrm{gr}(\mathrm{d}^\nabla)} = \mathrm{gr}(\mathrm{d}_N^\nabla)$ as a morphism

$$\partial^{\overline{\omega}} \colon \Lambda^k \mathfrak{t}_{H/H^0}^* M \otimes \mathrm{gr}(E) \to \Lambda^{k+1} \mathfrak{t}_{H/H^0}^* M \otimes \mathrm{gr}(E).$$

Proposition 5.6.2. *For every $x \in M$,*

$$(\Lambda^\bullet \mathfrak{t}_{H/H^0,x}^* M \otimes \mathrm{gr}(E_x), \partial^{\overline{\omega_x}}),$$

is a Chevalley-Eilenberg type sequence of operators associated to the map $\overline{\omega}_x$.

Proof. By construction $\partial^{\overline{\omega}}$ is the unique extension to higher exterior powers of the action of ω_x hence it is the Chevalley-Eilenberg differential obtained from this map. □

We did not call this sequence of operators a complex because it is a complex if and only if $\overline{\omega_x}$ is a representation of $\mathfrak{t}_{H/H^0,x}M$ on $\mathrm{gr}(E_x)$. This leads us to the necessary condition for the sequence to be transversally graded Rockland: the sequence of symbols has to be exact.

Proposition 5.6.3. *For each $x \in M$ we have the equivalence:*

a) The (2,0)-part of the curvature[8], $(R^\nabla)^{(2,0)} \in \Omega^{2,0}(M, \mathrm{End}(E))$ *has degree* 1, *i.e. if $x \in M$, $X \in H_x^i, Y \in H_x^j, \psi \in E_x^p$ then*

$$R^\nabla(X,Y)\psi \in E_x^{p+i+j+1}.$$

b) $\tilde{\sigma}_x^{0,\perp}(\mathrm{d}_N^\nabla)^2 = 0.$

c) $(\partial^{\overline{\omega_x}})^2 = 0.$

[8] We only showed that the curvature had this bi-degree for transverse parabolic geometries. This does not seem to hold in general.

d) $\overline{\omega_x}$ *is a representation of* $\mathfrak{t}_{H/H^0,x}M$ *on* $\mathrm{gr}(E_x)$ *that preserves the graduation.*

Proof. To make the link with the curvature recall that

$$(\mathrm{d}_N^\nabla)^2 = (R^\nabla)^{(2,0)} \wedge \cdot - (\mathrm{d}_F^\nabla \iota_\Theta + \iota_\Theta \, \mathrm{d}_F^\nabla).$$

If we show that the graded principal symbol of $(\mathrm{d}_F^\nabla \iota_\Theta + \iota_\Theta \, \mathrm{d}_F^\nabla)$ vanishes then we get the equivalence between $a), b)$ and $c)$ by the previous lemma and proposition. The equivalence $c) \Leftrightarrow d)$ is immediate:

$$(\partial^{\overline{\omega}_x})^2(X,Y) = [\omega(X), \omega(Y)] - \omega([X,Y]).$$

Now since ∇ preserves the filtration, elements of degree 0 do not change the degree. However Θ takes value in sections of the foliations, thus in elements of degree 0. Therefore the terms other that $(R^\nabla)^{(2,0)}$ vanish when we take the associated graded morphism and the transverse symbol. $\square$

Under these conditions we can state the main result:

Theorem 5.6.4. *Let* (M,H) *be a foliated filtered manifold* $E \to M$ *a filtered vector bundle and* ∇ *a* $\mathrm{Hol}(H^0)$*-equivariant connection on* E *that preserves the filtration and such that the* $(2,0)$*-component of its curvature has degree* 1*. Then the sequence of operators given by the transverse de Rham sequence* $(\Omega^{\bullet,0}(M,E), \mathrm{d}_N^\nabla)$ *is transversally graded Rockland.*

Proof. We have already computed the graded transverse symbol of d_N^∇. Under those assumptions we get $\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla)^2 = 0$. Let $x \in M$ and take a non-trivial irreducible unitary representation of the fiber group $\pi \in \widehat{T_{H/H^0,x}M} \setminus \{1\}$. The symbolic complex $(\Lambda^\bullet \mathfrak{t}^*_{H/H^0,x}M \otimes E_x \otimes \mathcal{H}_\pi^\infty, \mathrm{d}\pi(\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla)))$ is the Chevalley-Eilenberg complex of the $\mathfrak{t}_{H/H^0,x}M$-module $E_x \otimes \mathcal{H}_\pi^\infty$. We need to show that it is an exact sequence, i.e. that its cohomology is trivial. $\square$

Lemma 5.6.5. *Let* N *be a finite dimensional, connected, simply connected, nilpotent Lie group and* $\mathfrak{n}$ *its Lie algebra. Let* $\pi \in \hat{N} \setminus \{1\}$ *be a non-trivial irreducible unitary representation and* V *be a finite dimensional unitary representation of* N*. Then the associated Chevalley-Eilenberg cohomology vanishes:* $H^*(\mathfrak{n}, \mathcal{H}_\pi^\infty \otimes V) = 0$*.*

Proof. We first reduce to the case $V = 1$. Since $\mathfrak{n}$ is nilpotent, using Engel's theorem we can find an invariant subspace $W \subset V$ of dimension 1. The action on W is then trivial and the long exact sequence in cohomology becomes

$$\cdots \longrightarrow H^k(\mathfrak{n}, \mathcal{H}_\pi^\infty \otimes V) \longrightarrow H^k(\mathfrak{n}, \mathcal{H}_\pi^\infty \otimes {}^{V}\!/_{W}) \longrightarrow H^{k+1}(\mathfrak{n}, \mathcal{H}_\pi^\infty) \longrightarrow \cdots$$

It thus suffices to prove the result for $V = 1$ and we get every finite dimensional representations by induction. Now for the trivial representation we use induction on the dimension of $\mathfrak{n}$. Since $\mathfrak{n}$ is nilpotent it has a non-trivial center [53], we take $\mathfrak{z} \subset \mathfrak{n}$ a 1-dimensional central subalgebra. Now the Hochschild-Serre exact sequence [46] has its E_2-page equal to

$$E_2^{p,q} = H^p({}^{\mathfrak{n}}\!/\!_{\mathfrak{z}}, H^q(\mathfrak{z}, \mathcal{H}_\pi^\infty))$$

and converges to $H^*(\mathfrak{n}, \mathcal{H}_\pi^\infty)$. Now because π is an irreducible representation of G the action of $\mathfrak{z}$ on $\mathcal{H}_\pi^\infty$ is by scalar operators (the corresponding scalar is called the infinitesimal character, see [53]). If the action of $\mathfrak{z}$ is non-trivial then the cohomology vanishes. Indeed, $H^0(\mathfrak{z}, \mathcal{H}_\pi^\infty)$ is the space of vectors that vanish under the action of $\mathfrak{z}$ hence $\{0\}$. The space $H^1(\mathfrak{z}, \mathcal{H}_\pi^\infty)$ is the space of $\mathcal{H}_\pi^\infty$-valued outer derivations and is thus trivial (if D is such a derivation, we can write $Dx = \dfrac{1}{\lambda}\mathrm{d}\pi(Z)(Dx)$ where $Z \in \mathfrak{z} \setminus \{0\}$ satisfies $\mathrm{d}\pi(Z) = \lambda\,\mathrm{Id}_{\mathcal{H}_\pi^\infty}$ thus D is inner). In the trivial case we have

$$H^*(\mathfrak{z}, \mathcal{H}_\pi^\infty) = \mathcal{H}_\pi^\infty \oplus \mathcal{H}_\pi^\infty$$

as a ${}^{\mathfrak{n}}\!/\!_{\mathfrak{z}}$-module. To proceed with the induction, denote by $Z \subset N$ the closed subgroup of N with Lie algebra N. The group Z acts trivially on $\mathcal{H}_\pi^\infty$ and thus on $\mathcal{H}$ by continuity. The representation π factors to a non-trivial irreducible representation of ${}^{G}\!/\!_{Z}$ whose space of smooth vector fields is $\mathcal{H}_\pi^\infty$. By induction on the dimension of $\mathfrak{n}$ we have $H^*({}^{\mathfrak{n}}\!/\!_{\mathfrak{z}}, \mathcal{H}_\pi^\infty) = 0$ and thus the same result for $\mathfrak{n}$ using the spectral sequence. Indeed the E_2-page becomes either directly 0 or $H^p({}^{\mathfrak{n}}\!/\!_{\mathfrak{z}}, \mathcal{H}_\pi^\infty))$ for $q = 0, 1$ and thus still 0, the E_2 page vanishes and so does the cohomology $H^*(\mathfrak{n}, \mathcal{H}_\pi^\infty))$. □

5.6.2 Transverse parabolic geometries

As a particular case of theorem 5.6.4 we get:

Proposition 5.6.6. *Let M be a foliated manifold with transverse parabolic (G, P)-geometry. Let $\mathbb{E}$ be a G-representation and $E \to M$ the associated tractor bundle with the associated tractor connection ∇. If the geometry is regular then the transverse complex $(\Omega^{\bullet,0}(M, E), \mathrm{d}_N^\nabla)$ is transversally graded Rockland.*

Proof. The condition on the tractor connection and its curvature are consequences of propositions 5.4.5.1 and 5.2.9 respectively. □

We now want to study in more depth this sequence of operators and define an analog of the BGG sequences using the appropriate version of a transverse laplacian. This could be done in more generality with more hypotheses that would automatically be satisfied for regular transverse parabolic geometries. We refer to [27] where this general machinery is described in the non-foliated case. Other examples could include a transverse version of the Rumin complex and its corresponding hypoelliptic laplacian [74, 75].

To do this recall there is the co-differential on $\Lambda^\bullet\left({\mathfrak{g}}/{\mathfrak{p}}\right)^*$ inherited from the isomorphism of $\mathfrak{p}$-modules between $\left({\mathfrak{g}}/{\mathfrak{p}}\right)^*$ and $\mathfrak{p}_+$. Using the isomorphism ${TM}/{H^0} \cong M \times {\mathfrak{g}}/{\mathfrak{p}}$ given by the Cartan connection we can define a bundle map called the Kostant co-differential:

$$\partial^*\colon \Lambda^{\bullet,0}T^*M \otimes E \to \Lambda^{\bullet-1,0}T^*M \otimes E.$$

The algebraic co-differential was P-equivariant hence ∂^* is P-equivariant. Since the identification between ${TM}/{H^0}$ and $M \times {\mathfrak{g}}/{\mathfrak{p}}$ is done using the Cartan connection ω the map ∂^* is also $\mathrm{Hol}(\ker(\omega))$-equivariant and thus ∂^* is $\mathrm{Hol}(H^0)$-equivariant.

We now consider the differential operators of graded order 0:

$$\Box_\bullet := \mathrm{d}_N^\nabla\partial^* + \partial^*\,\mathrm{d}_N^\nabla\colon \Omega^{\bullet,0}(M,E) \to \Omega^{\bullet,0}(M,E).$$

Since $\overline{\mathrm{gr}(\mathrm{d}_N^\nabla)} = \partial_{\mathfrak{g}_-}$ we get that $\forall x \in M, \overline{\mathrm{gr}(\Box_\bullet)}_x = \Box_{\mathfrak{g}_-,\bullet}$ where $\Box_{\mathfrak{g}_-,\bullet}$ is Kostant's laplacian. Remember from section 5.1 that we have a Hodge decomposition and we can transpose it to the bundle level. Denote by

$$\widetilde{P}_k \in \mathrm{End}(\Lambda^k\mathfrak{t}^*_{H/H^0}M \otimes \mathrm{gr}(E)),$$

the projection onto the generalised zero eigenspace of $\overline{\mathrm{gr}(\Box_k)}$. On each fiber $\tilde{P}_k$ corresponds to the same type of projection constructed at the algebraic level. In particular $\tilde{P}_k$ is $\mathrm{Hol}(H^0)$-equivariant.

Remark 5.6.7. As described earlier, the graduation on $\mathbb{E}$ comes from the action of a particular element of $\mathfrak{g}_0$ (which was unique) and $\mathbb{E}$ itself is written as a graded sum of subspaces. Consequently this grading is G_0-equivariant, where $G_0 \subset G$ integrates $\mathfrak{g}_0$. Moreover we have $G_0 = {P}/{P_+}$ thus P preserves the filtration on $\mathbb{E}$ and acts trivially on the associated graded $\mathrm{gr}(\mathbb{E})$. Consequently we have $\mathrm{gr}(E) \cong \mathbb{E}$ as P-representations. Therefore at the bundle level we have $\mathrm{gr}(E) = E$ as $\mathrm{Hol}(H^0)$-equivariant bundles. In the case of regular transverse parabolic geometries we will therefore write E instead of $\mathrm{gr}(E)$ for a tractor bundle.

The homology spaces $H_*(\mathfrak{p}_+, \mathbb{E})$ are $\mathfrak{g}_0$-modules. We extend their structure to $\mathfrak{p}$-modules with a trivial action of $\mathfrak{p}_+$. We can thus define

$$\mathcal{H}_* := M \times H_*(\mathfrak{p}_+, \mathbb{E}) \to M,$$

as a $\mathrm{Hol}(H^0)$-equivariant bundle. Using Kostant's co-differential we get natural $\mathrm{Hol}(H^0)$-equivariant bundle maps $\pi\colon \ker(\partial_k^*) \to \mathcal{H}_k$. We can use the same arguments as in [27] to construct differential operators

$$P_\bullet\colon \Omega^{\bullet,0}(M, E) \to \Omega^{\bullet,0}(M, E).$$

They satisfy the following properties: $P_\bullet^2 = P_\bullet, P_\bullet\Box_\bullet = \Box_\bullet P_\bullet$, $P_\bullet$ preserves the filtration of the corresponding bundle and $\mathrm{gr}(P_\bullet) = \tilde{P}_\bullet$. By construction of $\tilde{P}_\bullet$ we have that $\Box_\bullet$ is nilpotent on $\mathrm{im}(P_\bullet)$ and invertible on $\ker(P_\bullet)$. Choose a splitting of $\mathfrak{t}_{H/H^0}M$. This induces splittings $S_\bullet$ of all the bundles $\Lambda^\bullet\mathfrak{t}^*_{H/H^0}M \otimes E$. We can now relate the decompositions induced by $P_\bullet$ and $\tilde{P}_\bullet$ with the operators:

$$L_\bullet = P_\bullet S_\bullet \tilde{P}_\bullet + (1 - P_\bullet)S_\bullet(1 - \tilde{P}_\bullet)\colon \Gamma(\Lambda^\bullet\mathfrak{t}^*_{H/H^0}M \otimes E) \xrightarrow{\sim} \Omega^{\bullet,0}(M, E).$$

These operators preserve the decompositions induced by $P_\bullet$ and $\tilde{P}_\bullet$ at the level of sections. They thus induce differential operators:

$$\bar{L}_\bullet\colon \Gamma(\mathcal{H}_\bullet) \to \mathrm{im}(P_\bullet),$$

with $\bar{L}_\bullet\pi_\bullet = P_\bullet$ and $\pi_\bullet\bar{L}_\bullet = 1$ where $\pi_\bullet$ are the natural quotient maps onto $\mathcal{H}_\bullet$.

Definition 5.6.8. The transverse BGG operators are the differential operators of graded filtered order 0:

$$D_\bullet := \pi_{\bullet+1}P_{\bullet+1}\, \mathrm{d}_N^\nabla L_\bullet\colon \mathcal{H}_\bullet \to \mathcal{H}_\bullet.$$

Theorem 5.6.9. *The BGG sequence $(\mathcal{H}_\bullet, D_\bullet)$ is transversally graded Rockland. Moreover if $(\Omega^{\bullet,0}(M, E), \mathrm{d}_N^\nabla)$ is a complex*[9] *then so is $(\mathcal{H}_\bullet, D_\bullet)$. The operators $\bar{L}_\bullet$ then define a chain map and induces an isomorphism between the cohomology of both complexes.*

Proof. Since $\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla)^2 = 0$ then we have:

$$\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla)\tilde{\sigma}^{0,\perp}(\Box_\bullet) = \tilde{\sigma}^{0,\perp}(\Box_{\bullet+1})\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla).$$

[9]This is the case for instance if there is an involutive complementary subbundle $N \subset TM$ to the foliation and the curvature K vanishes.

Therefore by construction of the projectors we also get the commutation relations $\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla)\tilde{\sigma}^{0,\perp}(P_\bullet) = \tilde{\sigma}^{0,\perp}(P_{\bullet+1})\tilde{\sigma}^{0,\perp}(\mathrm{d}_N^\nabla)$. In particular at the symbolic level the symbols $\tilde{\sigma}^{0,\perp}(L_{\bullet+1}^{-1}\,\mathrm{d}_N^\nabla L_\bullet)$ are diagonal with respect to the decomposition $\Gamma(\mathrm{im}(\tilde{P}_\bullet)) \oplus \Gamma(\ker(\tilde{P}_\bullet))$. Under the isomorphism $\Gamma(\mathcal{H}_\bullet) \cong \Gamma(\mathrm{im}(\tilde{P}_\bullet))$, $\tilde{\sigma}^{0,\perp}(D_\bullet)$ appears as the one of the two diagonal terms of $\tilde{\sigma}^{0,\perp}(L_{\bullet+1}^{-1}\,\mathrm{d}_N^\nabla L_\bullet)$. Therefore the BGG sequence $(\mathcal{H}_\bullet, D_\bullet)$ is also transversally graded Rockland. If $(\Omega^{\bullet,0}(M,E), \mathrm{d}_N^\nabla)$ then by construction so is $(\mathcal{H}_\bullet, D_\bullet)$. Moreover in this case the operator d_N^∇ itself is diagonal (using the same arguments as before). We have

$$L_{\bullet+1}^{-1}\,\mathrm{d}_N^\nabla L_\bullet = \tilde{D}_\bullet \oplus A_\bullet,$$

and both $(\Gamma(\mathrm{im}(\tilde{P}_\bullet)), \tilde{D}_\bullet)$ and $(\Gamma(\ker(\tilde{P}_\bullet)), A_\bullet)$ are complexes. Moreover $(\Gamma(\mathrm{im}(\tilde{P}_\bullet)), \tilde{D}_\bullet)$ is isomorphic to the BGG complex so we need to show that the chain map $L_\bullet$ between $(\Gamma(\mathrm{im}(\tilde{P}_\bullet)), \tilde{D}_\bullet)$ and $(\Omega^{\bullet,0}(M,E), \mathrm{d}_N^\nabla)$ is invertible up to homotopy. Define

$$G_\bullet = A_{\bullet-1}(1-\tilde{P_{\bullet-1}})\,\mathrm{gr}(\partial_\bullet^*)\,\mathrm{gr}(\Box_\bullet)^{-1} + (1-\tilde{P}_\bullet)\,\mathrm{gr}(\partial_{\bullet+1}^*)\,\mathrm{gr}(\Box_{\bullet+1})^{-1}\,\mathrm{gr}(\mathrm{d}_N^\nabla)$$

as a differential operator of graded filtered order 0 on $\ker(\tilde{P}_\bullet)$. We have $\mathrm{gr}(G_\bullet) = 1$ and is thus invertible, it conjugates the complex $(\Gamma(\ker(\tilde{P}_\bullet)), A_\bullet)$ to the acyclic complex $(\Gamma(\ker(\tilde{P}_\bullet)), \mathrm{gr}(\mathrm{d}_N^\nabla))$. This in itself gives the isomorphism in cohomology. We can also write an explicit homotopy. The operator

$$h_\bullet := L_{\bullet-1}G_{\bullet-1}(1-\tilde{P_{\bullet-1}})\,\mathrm{gr}(\partial_\bullet^*)\,\mathrm{gr}(\Box_\bullet)^{-1}G_\bullet^{-1}(1-\tilde{P}_\bullet)L_\bullet^{-1},$$

maps $\Omega^{\bullet,0}(M,E)$ to $\Omega^{\bullet-1,0}(M,E)$ and satisfies

$$1 - L_\bullet\tilde{P}_\bullet L_\bullet^{-1} = \mathrm{d}_N^\nabla h_\bullet + h_{\bullet+1}\,\mathrm{d}_N^\nabla.$$

Since $\tilde{P}_\bullet L_\bullet^{-1} L_{\bullet|\Gamma(\mathrm{im}(\tilde{P}_\bullet))} = 1$ and $\tilde{D}_\bullet\tilde{P}_\bullet L_\bullet^{-1} = \tilde{P}_\bullet L_\bullet^{-1}\,\mathrm{d}_N^\nabla$ we get that $\tilde{P}_\bullet L_\bullet^{-1}$ is the inverse of $L_\bullet$ up to homotopy. □

Chapter 6

Conclusion and perspectives

In this thesis we have extended results of the classical pseudodifferential calculus to the calculus of filtered manifolds. We have also added foliations to the setting of filtered manifolds to consider filtrations of the space of leaves. However the foliations considered were regular (i.e. given by an integrable subbundle of the tangent bundle). The main tool was the holonomy groupoid and its construction can be extended to singular foliations following [3]. However if we consider singular foliations, the filtration should be extended to a singular version as well. This was done recently by Androulidakis, Mohsen and Yuncken [2] in their proof of the Helffer-Nourrigat conjecture. It would be interested to extend the work of chapter 4 to the singular case. In this way we could understand foliated filtrations where the foliation is given by the action of a Lie group (not necessarily locally free). An outcome of such an extension could be a formulation of a "quantization commutes with reduction" type problem for filtered manifolds, especially the contact ones.

Another idea would be to apply the ideas of calculus on filtered manifolds to concrete geometric examples. An interesting one is given by Poisson geometry. To define a non-formal star-product on symplectic manifolds and extend a construction of Guillemin [40], Melrose used the Heisenberg calculus of a particular contact manifold [62]. Indeed in the prequantizable case, the total space of the underlying circle bundle is a contact manifold (there is a similar idea in the non-prequantizable case). For more general Poisson manifolds one would have to first integrate it to a symplectic groupoid. The corresponding contact manifold should thus be a contact groupoid in the sense of [51, 28]. On such a manifold the algebroid is filtered so the techniques used in this thesis could apply.

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